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JUNCTION OF MODELS OF DIFFERENT DIMENSION FOR FLOWS IN TUBE STRUCTURES BY WOMERSLEY-TYPE INTERFACE CONDITIONS*

CRISTÓBAL BERTOGLIO[†], CARLOS CONCA[‡], DAVID NOLTE^{†‡}, GRIGORY PANASENKO[§], AND KONSTANTINAS PILECKAS[¶]

Abstract. The method of asymptotic partial decomposition of a domain proposed and justified earlier for thin domains (rod structures, tube structures consisting of a set of thin cylinders) generates some special interface conditions between the three-dimensional and one-dimensional parts. In the case of fluid mechanics these conditions prescribe a precomputed Poiseuille-type shape of a solution at the interface, which, however, are not generalizable to the case with a boundary layer in time. In this work we present a new more general version of the method which considered and justified the transient Navier–Stokes equations. Although theoretical justification (well posedness, asymptotic analysis) can be shown only for moderate Reynolds numbers, the provided numerical tests show good accuracies for higher values.

Key words. Stokes equations, Navier–Stokes equations, thin structures, asymptotic partial decomposition, hybrid dimension models

AMS subject classifications. 35Q35, 76D07, 65N55

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1. Introduction. The Stokes and Navier–Stokes equations in thin tube structures are the most classical models for a viscous flow in pipelines or blood vessels. Tube structures are domains which are tree-like sets of thin cylinders (or thin rectangles in a two-dimensional setting). The ratio of the diameters of cylinders to their heights (or ratio of the sides of rectangles) is a small parameter ε . The method of asymptotic partial decomposition of a domain (MAPDD) allows one to reduce essentially the computer resources needed for the numerical solution of such problems. This method combines the full-dimensional description in some neighborhoods of bifurcations and a reduced-dimensional description out of these small subdomains and it prescribes some special junction conditions at the interface between these three-dimensional and one-dimensional submodels (see [6, 13, 19, 15]). In particular, for the nonsteady Navier–Stokes equations these interface conditions prescribe a precomputed Poiseuille-type shape. To this end one has to solve a Jordanian chain of elliptic equations on the section and take their linear combination [15]. This condition is justified for the

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Navier–Stokes equation without a boundary layer in time, when the right-hand side of the boundary condition vanishes for small values of time. However, in the case of a general setting the question on the high order interface conditions is still open [16]. The goal of the paper is to give and justify a more general interface condition which is applicable for the problems with a boundary layer in time. Such a condition is constructed for the steady state Stokes equations and then is generalized for the nonstationary Navier–Stokes equations. In this new version the trial and test functions have vanishing transversal components of the velocity and vanishing normal derivative of the normal component inside the cylinders, instead of the precomputed Poiseuille-type shape. This also leads to an easy-to-implement finite element formulation of the MAPDD and to assessing it numerically in dependence of the Reynolds number.

The remainder of paper is organized as follows. In section 2 the full-dimensional Dirichlet’s problem for the nonstationary Navier–Stokes equations and stationary Stokes equations in a thin tube structure are formulated. We give two weak formulations: one containing only the unknown velocity (formulation “without pressure” which is convenient for the asymptotic analysis) and one formulation containing both unknown velocity and unknown pressure which is convenient for the numerical solution. In section 3 the original MAPDD method is revisited. In section 4 the new version of MAPDD for the steady Stokes and transient Navier–Stokes equations is introduced and the main theorems summarized. For the sake of readability by a wide range of specialists the proofs are moved to the appendices. Finally, we present some numerical examples in section 5, where the theoretical results are confirmed. Note that recently an asymptotic analysis of flows of complex rheology in thin tube structures was developed in [8, 3].

2. The full-dimensional fluid flow problem in a tube structure. In this section we will introduce the full-dimensional fluid flow problem in a tube structure. Further, its solution will be approximated using partial dimension reduction.

2.1. Thin tube structure domain. Let us recall the definition of a thin tube structure [18, 20, 15], graphically exemplified in Figure 2.1.

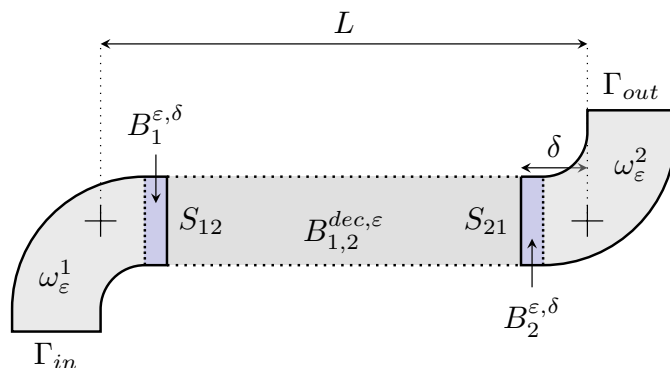


FIG. 2.1. Illustration of the computational domain for $N = 2$ and $M = 1$.

Let O_1, O_2, \dots, O_N be N different points in $\mathbb{R}^n, n = 2, 3$, and e_1, e_2, \dots, e_M be M closed segments each connecting two of these points (i.e., each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \dots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j . The segments e_j are called edges of the graph. The points O_i are called

nodes. Any two edges e_j and e_i , $i \neq j$, can intersect only at the common node. A node is called a vertex if it is an end point of only one edge. Assume that the set of vertices is $O_{N_1+1}, O_{N_1+2}, \dots, O_N$, where $N_1 < N$. Denote $\mathcal{B} = \bigcup_{j=1}^M e_j$ the union of edges, and assume that \mathcal{B} is a connected set. The graph \mathcal{G} is defined as the collection of nodes and edges. Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in \mathbb{R}^n . The first one has the origin in O_i and the axis $O_i x_1^{(e)}$ has the direction of the ray $[O_i O_j]$; the second one has the origin in O_j and the opposite direction, i.e., $O_j \tilde{x}_1^{(e)}$ is directed over the ray $[O_j O_i]$. With every edge e_j we associate a bounded domain $\sigma_j \subset \mathbb{R}^{n-1}$ having a C^2 -smooth boundary $\partial\sigma^j$, $j = 1, \dots, M$. For every edge $e_j = e$ and associated $\sigma_j = \sigma^{(e)}$ we denote by $B_\varepsilon^{(e)}$ the cylinder

$$B_\varepsilon^{(e)} = \left\{ x^{(e)} \in \mathbb{R}^n : x_1^{(e)} \in (0, |e|), \frac{x^{(e)'} }{\varepsilon} \in \sigma^{(e)} \right\},$$

where $x^{(e)'} = (x_2^{(e)}, \dots, x_n^{(e)})$, $|e|$ is the length of the edge e , and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j , as well as the domains σ_j , do not depend on ε . Denoting $\sigma_\varepsilon^{(e)} = \{x^{(e)'} \in \mathbb{R}^{n-1} : \frac{x^{(e)'} }{\varepsilon} \in \sigma^{(e)}\}$ we can write $B_\varepsilon^{(e)} = (0, |e|) \times \sigma_\varepsilon^{(e)}$. Let $\omega^1, \dots, \omega^N$ be bounded independent of ε domains in \mathbb{R}^n with Lipschitz boundaries $\partial\omega^j$; we introduce the nodal domains: $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \frac{x - O_j}{\varepsilon} \in \omega^j\}$. Denote $d = \max_{1 \leq j \leq N} \text{diam } \omega^j$. By a tube structure we call the following domain:

$$B_\varepsilon = \left(\bigcup_{j=1}^M B_\varepsilon^{(e_j)} \right) \cup \left(\bigcup_{j=1}^N \omega_\varepsilon^j \right).$$

So, the tube structure B_ε is a union of all thin cylinders having edges as the heights plus small smoothing domains ω_ε^j in the neighborhoods of the nodes. Their role is to avoid artificial corners in the boundary of intersecting cylinders, and we will assume that B_ε is a bounded domain (connected open set) with a C^2 -smooth boundary. However, for the numerical tests we consider a domain with corners.

2.2. The full-dimensional fluid flow problem. Throughout the paper we will consider the stationary Stokes or the nonstationary Navier–Stokes equations in B_ε with the no-slip conditions at the boundary ∂B_ε except for some parts γ_ε^j of the boundary where the velocity field is given as known inflows and outflows (for alternative boundary conditions on the inlet and outlet boundaries of the domain, the reader is referred to [2, 5]).

Let us define these parts of the boundary. Denote $\gamma_\varepsilon^j = \partial\omega_\varepsilon^j \cap \partial B_\varepsilon$, $\gamma^j = \partial\omega^j \cap \partial B_1^j$, where $B_1^j = \{y : y\varepsilon + O_j \in B_\varepsilon\}$ and $\gamma_\varepsilon = \bigcup_{j=N_1+1}^N \gamma_\varepsilon^j$.

Let us introduce first the initial boundary value problem for the nonstationary Navier–Stokes equations,

$$\begin{aligned} (2.1) \quad & \mathbf{u}_\varepsilon t - \nu \Delta \mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0, \\ & \text{div } \mathbf{u}_\varepsilon = 0, \\ & \mathbf{u}_\varepsilon|_{\partial B_\varepsilon} = \mathbf{g}_\varepsilon, \quad \mathbf{u}_\varepsilon(x, 0) = 0, \end{aligned}$$

where \mathbf{u}_ε is the unknown velocity vector, p_ε is the unknown pressure, and \mathbf{g}_ε is a given vector-valued function satisfying the conditions $\mathbf{g}_\varepsilon(x, t) = \mathbf{g}_j(\frac{x - O_j}{\varepsilon}, t)$ if $x \in \gamma_\varepsilon^j$, $j = N_1 + 1, \dots, N$, and equal to zero for the remaining part of the boundary $\partial B_\varepsilon \setminus \gamma_\varepsilon$. Here

$\mathbf{g}_j : \gamma_j \times [0, +\infty) \rightarrow \mathbb{R}^n$ belonging to $C^{[\frac{j+4}{2}]+1}([0, T]; \mathbf{H}_0^{3/2}(\gamma_j))$, and T is a positive number. Assume that $\mathbf{g}_j|_{t=0} = 0$ and (the compatibility condition)

$$(2.2) \quad \int_{\partial B_\varepsilon} \mathbf{g}_\varepsilon \cdot \mathbf{n} ds = \sum_{j=N_1+1}^N \int_{\gamma_\varepsilon^j} \mathbf{g}_j \left(\frac{x - O_j}{\varepsilon}, t \right) \cdot \mathbf{n} ds = 0.$$

Remark 2.1. In this case one can prove that \mathbf{g}_ε has a divergence-free extension $\tilde{\mathbf{g}}$ defined in $B_\varepsilon \times [0, T]$ which we denote by the same symbol \mathbf{g}_ε , $\mathbf{g}_\varepsilon \in C^{[\frac{j+4}{2}]+1}([0, T]; H^2(B_\varepsilon))$ satisfying for all $t \in [0, T]$ the following asymptotic estimates:

$$(2.3) \quad \begin{aligned} \|\mathbf{g}_\varepsilon\|_{L^2(B_\varepsilon)} + \|\mathbf{g}_{\varepsilon,t}\|_{L^2(B_\varepsilon)} + \|\mathbf{g}_{\varepsilon,tt}\|_{L^2(B_\varepsilon)} &\leq c\varepsilon^{\frac{n-1}{2}}; \\ \|\nabla \mathbf{g}_\varepsilon\|_{L^2(B_\varepsilon)} + \|\nabla \mathbf{g}_{\varepsilon,t}\|_{L^2(B_\varepsilon)} &\leq c\varepsilon^{\frac{n-3}{2}}, \\ \|\Delta \mathbf{g}_\varepsilon\|_{L^2(B_\varepsilon)} &\leq c\varepsilon^{\frac{n-5}{2}}, \quad n = 2, 3, \end{aligned}$$

where the constant c is independent of ε (see [15], [16]).

There are two equivalent weak formulations of the problem, “with pressure” and “without pressure,” which differ by the space of test functions. In the formulation “without pressure” test functions are divergence free and so the integral containing the pressure disappears; the only unknown function is the vector of velocity. In the formulation “with pressure” the space of test functions is wider, and they may not be divergence free, so that the pressure participates in the formulation as an unknown function. The formulation “without pressure” is used mainly in analysis, while the definition “with pressure” is more convenient for the numerical approximation using finite elements because it doesn’t require construction of divergence-free bases in the space of test functions.

We introduce the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$ as the subspace of vector-valued functions from $\mathbf{H}^1(B_\varepsilon)$ satisfying the conditions $\operatorname{div} \mathbf{v} = 0$, $\mathbf{v}|_{\partial B_\varepsilon \setminus \gamma_\varepsilon} = 0$, i.e.,

$$\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon) = \{ \mathbf{v} \in \mathbf{H}^1(B_\varepsilon) \mid \operatorname{div} \mathbf{v} = 0; \mathbf{v}|_{\partial B_\varepsilon \setminus \gamma_\varepsilon} = 0 \}.$$

We consider as well the smaller subspace $\mathbf{H}_{div0}^1(B_\varepsilon) = \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon) \cap \mathbf{H}_0^1(B_\varepsilon)$ of divergence-free vector-valued functions vanishing at the whole boundary.

DEFINITION 1. By a weak solution we understand the couple of the vector-field \mathbf{u}_ε and a scalar function p_ε such that $\mathbf{u}_\varepsilon(x, 0) = 0$, $\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon))$, $\mathbf{u}_{\varepsilon,t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon))$, $p_\varepsilon \in L^2(0, T; L^2(B_\varepsilon))$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε , and $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfy the integral identity for every vector-field $\phi \in \mathbf{H}_0^1(B_\varepsilon)$ for all $t \in (0, T)$,

$$(2.4) \quad \int_{B_\varepsilon} (\mathbf{u}_{\varepsilon,t} \cdot \phi + \nu \nabla \mathbf{u}_\varepsilon : \nabla \phi + ((\mathbf{u}_\varepsilon, \nabla \mathbf{u}_\varepsilon) \cdot \phi)) dx = \int_{B_\varepsilon} p_\varepsilon \operatorname{div} \phi dx.$$

Replacing the space of test functions by a subspace of divergence-free functions we get another weak formulation without the integral $\int_{B_\varepsilon} p_\varepsilon \operatorname{div} \phi dx$.

DEFINITION 2. By a weak solution we understand the vector-field \mathbf{u}_ε such that $\mathbf{u}_\varepsilon(x, 0) = 0$, $\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon))$, $\mathbf{u}_{\varepsilon,t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon))$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε , and \mathbf{u}_ε satisfies the integral identity for every vector-field $\phi \in \mathbf{H}_{div0}^1(B_\varepsilon)$ for all $t \in (0, T)$,

$$(2.5) \quad \int_{B_\varepsilon} (\mathbf{u}_{\varepsilon,t} \cdot \phi + \nu \nabla \mathbf{u}_\varepsilon : \nabla \phi + ((\mathbf{u}_\varepsilon, \nabla \mathbf{u}_\varepsilon) \cdot \phi)) dx = 0.$$

For sufficiently small ε there exists a unique solution to this problem (see [15]). The equivalence of these formulations follows from [11]; see also [23].

Consider the Dirichlet's boundary value problem for the stationary Stokes equation,

$$(2.6) \quad \begin{aligned} -\nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon &= 0, \quad x \in B_\varepsilon, \\ \operatorname{div} \mathbf{u}_\varepsilon &= 0, \quad x \in B_\varepsilon, \\ \mathbf{u}_\varepsilon &= \mathbf{g}_\varepsilon, \quad x \in \partial(B_\varepsilon), \end{aligned}$$

where ν is a positive constant, and \mathbf{g}_ε is a given vector-valued function satisfying the conditions $\mathbf{g}_\varepsilon(x) = \mathbf{g}_j(\frac{x-O_j}{\varepsilon})$ if $x \in \gamma_\varepsilon^j, j = N_1+1, \dots, N$ (O_j are vertices!), and equal to zero for the remaining part of the boundary $\partial B_\varepsilon \setminus \gamma_\varepsilon$. Here $\mathbf{g}_j : \gamma_j \rightarrow \mathbb{R}^n$ belonging to $\mathbf{H}_0^{3/2}(\gamma_j)$. Assume that the compatibility condition (2.2) holds.

Remark 2.2. In the stationary case as well one can prove that \mathbf{g}_ε has a divergence-free extension $\tilde{\mathbf{g}}$ defined in B_ε which we denote by the same symbol \mathbf{g}_ε , $\mathbf{g}_\varepsilon \in \mathbf{H}^2(B_\varepsilon)$ (see Lemma A.1 in Appendix A).

Let us give two equivalent definitions of a weak solution. The first one is "with pressure."

DEFINITION 1'. By a weak solution we understand the couple of the vector-field \mathbf{u}_ε and a scalar function p_ε such that $\mathbf{u}_\varepsilon \in \mathbf{H}_{\operatorname{div}0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$, $p_\varepsilon \in L^2(B_\varepsilon)$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε , and $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfy the integral identity: for any test function $\mathbf{v} \in \mathbf{H}_0^1(B_\varepsilon)$

$$(2.7) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{u}_\varepsilon(x) : \nabla \mathbf{v}(x) dx = \int_{B_\varepsilon} p_\varepsilon \operatorname{div} \phi dx.$$

The second is "without pressure."

DEFINITION 2'. By a weak solution we understand the vector-field \mathbf{u}_ε such that $\mathbf{u}_\varepsilon \in \mathbf{H}_{\operatorname{div}0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$, $\mathbf{u}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε , and \mathbf{u}_ε satisfies the integral identity: for any test function $\mathbf{v} \in \mathbf{H}_{\operatorname{div}0}^1(B_\varepsilon)$

$$(2.8) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{u}_\varepsilon(x) : \nabla \mathbf{v}(x) dx = 0.$$

It is well known that there exists a unique solution to this problem (see [11]). The equivalence of these formulations follows from [11]; see also [23].

3. MAPDD: The classical version.

3.1. The reduced domain and classical version of MAPDD. Let us recall first the definition of the steady Poiseuille flow in a cylinder $B_\varepsilon^{(e)}$.

If the local variables $x^{(e)}$ for the edge e coincide with the global ones x , then the Poiseuille flow is defined as $\mathbf{V}_P^{(e)}(x) = \operatorname{const}(v_P(x'/\varepsilon), 0, \dots, 0)^T$, where $v_P(y)$ is a solution to the Dirichlet's problem for the Poisson equation on $\sigma^{(e)}$:

$$(3.1) \quad -\nu \Delta v_P(y) = 1, \quad y \in \sigma^{(e)}, \quad v_P(y) = 0, \quad y \in \partial \sigma^{(e)}.$$

If e has the cosines directors k_{e1}, \dots, k_{en} and the local variables $x^{(e)}$ are related to the global ones by equation $x^{(e)} = x^{(e)}(x)$, then the Poiseuille flow is

$$\mathbf{V}_P^{(e)}(x) = \operatorname{const}(k_{e1} v_P((x^{(e)}(x))'/\varepsilon), \dots, k_{en} v_P((x^{(e)}(x))'/\varepsilon))^T, \quad x' = (x_2, \dots, x_n).$$

In the case $\text{const} = 1$ denote it by $\mathbf{V}_P^{0,(e)}$; it is the normalized Poiseuille flow.

Let δ be a small positive number much greater than ε but much smaller than 1. For any edge $e = \overline{O_i O_j}$ of the graph introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends; see Figure 2.1.

Denote the cross sections of the cylinder $B_\varepsilon^{(e)}$ by these two hyperplanes, respectively, by $S_{i,j}$ (the cross section at the distance δ from O_i), and $S_{j,i}$ (the cross section at the distance δ from O_j), and denote the part of the cylinder between these two cross sections by $B_{ij}^{dec,\varepsilon}$. Denote $B_i^{\varepsilon,\delta}$ the connected truncated by the cross sections $S_{i,j}$, part of B_ε containing the vertex or the node O_i .

Define the subspace $\mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ (and, respectively, $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$) of the space $\mathbf{H}_{div0}^1(B_\varepsilon)$ (respectively, of $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$) such that on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ its elements described in local variables $x^{(e)}$ for the edge e (vector-valued functions) have a form of the Poiseuille flow $\mathbf{V}_P^{(e)}(x)$.

The MAPDD replaces the original full-dimensional problem for the steady Stokes equations (2.6) by the following weak formulation.

Find $\mathbf{u}_{\varepsilon,\delta} \in \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ such that $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε and for all $\mathbf{v} \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$,

$$(3.2) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{u}_{\varepsilon,\delta}(x) : \nabla \mathbf{v}(x) dx = 0.$$

For the nonstationary Navier–Stokes equations the Poiseuille flow has a more complicated structure [15]. For small ε it can be approximated by a time dependent linear combination of vector-valued functions $\mathbf{V}_{P,1}(x), \dots, \mathbf{V}_{P,J}(x)$ such that in local variables their first component $v_{P,j}(y)$ satisfies a Jordanian chain of equations

$$(3.3) \quad -\nu \Delta v_{P,j+1}(y) = -v_{P,j}(y), \quad y \in \sigma^{(e)}, \quad v_{P,j+1}(y) = 0, \quad y \in \partial \sigma^{(e)},$$

while the transversal components of vectors $\mathbf{V}_{P,1}(x), \dots, \mathbf{V}_{P,J}(x)$ are equal to zero, $\mathbf{V}_{P,1}(x) = \mathbf{V}_P(x)$ (the steady Poiseuille flow), and so the space of test functions for the MAPDD $\mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ is a subspace of $\mathbf{H}_{div0}^1(B_\varepsilon)$ such that on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ its elements described in local variables $x^{(e)}$ for the edge e (vector-valued functions) have a form of linear combinations of these functions $\alpha_1 \mathbf{V}_{P,1}(x) + \dots + \alpha_J \mathbf{V}_{P,J}(x)$, $\alpha_1, \dots, \alpha_J$ are real numbers.

Define the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ as a similar subspace of $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$. The weak formulation of the classical version of MAPDD for the nonstationary Navier–Stokes problem is given in [15]. It is equivalent to the following formulation without pressure: by a weak solution we understand the vector-field $\mathbf{u}_{\varepsilon,\delta}$ such that

$$\mathbf{u}_{\varepsilon,\delta}(x, 0) = 0, \mathbf{u}_{\varepsilon,\delta} \in L^\infty(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)), \mathbf{u}_{\varepsilon,\delta,t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon)),$$

$\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε , and $\mathbf{u}_{\varepsilon,\delta}$ satisfies the integral identity for every vector-field $\phi \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ for all $t \in (0, T)$,

$$(3.4) \quad \int_{B_\varepsilon} \left(\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + ((\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi) \right) dx = 0.$$

Existence and uniqueness of a solution for sufficiently small ε are proved as in [15] by the Galerkin method.

3.2. Summary of main results on the classical version. For the classical version of MAPDD the theorem on the error estimates is proved. Namely, it was proved that given J there exists a constant C independent of ε such that if $\delta = CJ\varepsilon|\ln(\varepsilon)|$, then for the Stokes equations the following estimate holds [19], [6]:

$$(3.5) \quad \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon,\delta}\|_{H^1(B_\varepsilon)} = O(\varepsilon^J).$$

For the nonstationary Navier–Stokes equations we have the following result [15].

Given natural number J , if $\mathbf{g}_j \in C^{[\frac{J+4}{2}]+1}([0, T]; W^{3/2,2}(\partial\omega_j))$ and there exists an interval $(0, \tau)$, $\tau > 0$ such that $\mathbf{g}_j = 0$ for $t \in (0, \tau)$, then there exists a constant C (independent of ε and J) such that if $\delta = CJ\varepsilon|\ln \varepsilon|$, then

$$(3.6) \quad \sup_{t \in (0, T)} \|\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)} + \|\nabla \mathbf{u}_{\varepsilon,\delta} - \nabla \mathbf{u}_\varepsilon\|_{L^2((0, T); L^2(B_\varepsilon))} = O(\varepsilon^J).$$

Although this classical version of the MAPDD is an effective method reducing considerably the computational costs, it does not work in the situation when the above condition $\mathbf{g}_j = 0$ for $t \in (0, \tau)$ is not satisfied. Indeed, in [16] it was shown that for small values of time of order ε^2 linear combinations of functions $\mathbf{V}_{P,i}$ are no longer a good approximation for the velocity inside the tubes; they should be replaced by the “boundary layer in time.” Moreover, the coordinate change from velocity degrees of freedom to $\alpha_1, \dots, \alpha_J$ involves intrusive modifications of the numerical simulation software, for both system assembly and linear algebra parts.

4. MAPDD: The new junction conditions. We now propose a new, more general, formulation of the method involving new junction conditions. The advantages are twofold: (1) it removes the condition $\mathbf{g}_j = 0$ for $t \in (0, \tau)$, therefore being applicable for arbitrary transient regimes, and (2) it considerably simplifies the numerical implementation in the context of finite elements since only additional, easy-to-build integral terms need to be added to a standard weak form.

4.1. Formulation of the new version. Let us define the subspace $\mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ (respectively, $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$) of the space $\mathbf{H}_{div0}^1(B_\varepsilon)$ (respectively, $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$) in a different way, so that on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ its elements described in local variables (vector-valued functions) have vanishing transversal (tangential) components while the longitudinal (normal) component has vanishing longitudinal (normal) derivative. Namely, if the local variables $x^{(e)}$ for the edge e coincide with the global ones x , then they have a form of the Womersley flow $\mathbf{W}_P^{(e)}(x) = (v_1(x'/\varepsilon), 0, \dots, 0)^T$, $v_1 \in H_0^1(\sigma^{(e)})$.

If e has the cosines directors k_{e1}, \dots, k_{en} and the local variables $x^{(e)}$ are related to the global ones by equation $x^{(e)} = x^{(e)}(x)$, then they are

$$\mathbf{W}_P^{(e)}(x) = \text{const}(k_{e1}v_1((x^{(e)}(x))'/\varepsilon), \dots, k_{en}v_1((x^{(e)}(x))'/\varepsilon))^T, x' = (x_2, \dots, x_n).$$

As in the classical version the MAPDD replaces the problem (2.6) by its projection on this newly defined space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$. Note that this space is wider than the space of test functions in the classical version because the steady Poiseuille flow is a particular case of functions $\mathbf{W}_P^{(e)}$. The weak formulations repeat literally the formulations of the previous section but with respect to the newly defined space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$.

4.2. Stokes equations.

Consider the Stokes equations (2.6). The new version of the MAPDD replaces the problem (2.6) by its projection on $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$: to find $\mathbf{u}_{\varepsilon,\delta} \in \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ such that $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε , and satisfies the following integral identity for all vector-fields for all $\mathbf{v} \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$:

$$(4.1) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{u}_{\varepsilon,\delta}(x) : \nabla \mathbf{v}(x) dx = 0.$$

Applying the Lax–Milgram argument one can prove that there exists a unique solution $\mathbf{u}_{\varepsilon,\delta}$ of the partially decomposed problem.

Remark 4.1. The classical version of MAPDD differs from this new one by the definition of the space on which we project the problem. Namely, in the new version the projection is taken onto the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ involving the Womersley functions, while in the classical case [20] it is a subspace of $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^1(B_\varepsilon)$ such that on every truncated cylinder $B_{ij}^{dec,\varepsilon}$ its elements are equal to a Poiseuille flow $\mathbf{V}_P^{(e)}$. Note that the space of Womersley functions is much wider than the space of Poiseuille flows.

4.3. Estimate for the difference between the exact solution and the MAPDD solution: Asymptotic analysis of the Stokes equations.

THEOREM 4.2. *Given natural number J there exists a constant C (independent of ε and J) such that if $\delta = CJ\varepsilon |\ln \varepsilon|$, then*

$$(4.2) \quad \|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon,\delta}\|_{\mathbf{H}^1(B_\varepsilon)} = O(\varepsilon^J) .$$

This estimate is the same as in the classical version of the MAPDD. The proof is similar to that of the classical version. However, for the sake of completeness we give it in Appendix A.2.

4.4. Navier–Stokes equations.

Consider the Navier–Stokes equations (2.1). The new version of the MAPDD replaces the problem (2.1) by (2.5), where the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ is replaced by the newly defined space of divergence-free vector-functions having the Womersley form within cylinders $B_{ij}^{dec,\varepsilon}$: by a weak solution we understand the vector-field $\mathbf{u}_{\varepsilon,\delta}$ such that $\mathbf{u}_{\varepsilon,\delta}(x, 0) = 0$, $\mathbf{u}_{\varepsilon,\delta} \in L^\infty(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon))$, $\mathbf{u}_{\varepsilon,\delta,t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon))$, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ on γ_ε , and $\mathbf{u}_{\varepsilon,\delta}$ satisfies the integral identity for every vector-field $\phi \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ for all $t \in (0, T)$,

$$(4.3) \quad \int_{B_\varepsilon} (\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + ((\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi)) dx = 0.$$

The existence and uniqueness of its solution are proved as in [15].

Let us give the formulation “with pressure.” Note that it is less evident than for the full-dimensional problem. First note that knowing the velocity field $\mathbf{u}_{\varepsilon,\delta}$, solution to problem (4.3), we can reconstitute some function $p_{\varepsilon,\delta}$ which is interpreted as the MAPDD pressure. Namely, let us denote $\mathbf{U}_{ij}(x^{(e)'}, t)$ the trace of the solution $\mathbf{u}_{\varepsilon,\delta}$ to problem (4.3) at every cross section S_{ij} . Then we get a standard Navier–Stokes problem in each domain $B_i^{\varepsilon,\delta}$ with the known boundary value $\mathbf{U}_{ij}(x^{(e)'}, t)$ on S_{ij} , the no-slip boundary condition on $\partial B_i^{\varepsilon,\delta} \setminus \Sigma_i$ if $i = 1, \dots, N_1$, or on $\partial B_i^{\varepsilon,\delta} \setminus (\Sigma_i \cup \gamma_i^\varepsilon)$ if

$i = N_1 + 1, \dots, N$ and respectively with condition $\mathbf{U}_{ij} = \mathbf{g}_\varepsilon$ at γ_i^ε in the last case; the initial condition is $\mathbf{U}_{ij}(x, 0) = 0$. Here Σ_i is a union $\cup_{j: \overline{O_i O_j} \in \{e_1, \dots, e_M\}} S_{ij}$ of all cross sections S_{ij} belonging to the boundary of $B_i^{\varepsilon, \delta}$. This problem admits a unique solution-velocity (coinciding with $\mathbf{u}_{\varepsilon, \delta}$) and a pressure $p_{\varepsilon, \delta, i}$ unique up to an additive function θ_i of t . Let us introduce an extended space of the test functions

$$\mathbf{H}_0^{1, \delta}(B_\varepsilon) = \left\{ \phi \in \mathbf{H}_0^1(B_\varepsilon) \mid \phi(x) = \mathbf{W}_\mathbf{P}^{(\mathbf{e})}(x), x \in B_{ij}^{dec, \varepsilon}, e = \overline{O_i O_j}; \int_{\partial B_i^{\varepsilon, \delta}} \phi \cdot \mathbf{n} = 0 \right\},$$

$i = 1, \dots, N$, and extend the integral identity (4.3) for test functions of this space:

$$\begin{aligned} & \int_{B_\varepsilon} (\mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon, \delta} : \nabla \phi + (\mathbf{u}_{\varepsilon, \delta}, \nabla \mathbf{u}_{\varepsilon, \delta}) \cdot \phi) dx \\ &= \sum_{i=1}^N \left(\int_{B_i^{\varepsilon, \delta}} (\mathbf{u}_{\varepsilon, \delta, t} \cdot \phi - \nu \Delta \mathbf{u}_{\varepsilon, \delta} + (\mathbf{u}_{\varepsilon, \delta}, \nabla \mathbf{u}_{\varepsilon, \delta})) \cdot \phi dx \right. \\ & \quad + \sum_{j: \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{\partial B_i^{\varepsilon, \delta} \cap S_{ij}} \nu \frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial \mathbf{n}} \phi ds \Big) \\ & \quad + \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla_{x^{(e_l)}, \prime} \mathbf{u}_{\varepsilon, \delta} : \nabla_{x^{(e_l)}, \prime} \phi dx^{(e_l), \prime} \\ &= \sum_{i=1}^N \left(- \int_{B_i^{\varepsilon, \delta}} \nabla p_{\varepsilon, \delta, i} \cdot \phi dx + \sum_{j: \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{\partial B_i^{\varepsilon, \delta} \cap S_{ij}} \nu \frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial \mathbf{n}} \cdot \phi ds \right) \\ & \quad + \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla_{x^{(e_l)}, \prime} \mathbf{u}_{\varepsilon, \delta} : \nabla_{x^{(e_l)}, \prime} \phi dx^{(e_l), \prime} \\ &= \sum_{i=1}^N \left(\int_{B_i^{\varepsilon, \delta}} p_{\varepsilon, \delta, i} \operatorname{div} \phi dx + \sum_{j: \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{\partial B_i^{\varepsilon, \delta} \cap S_{ij}} \left(\nu \frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial \mathbf{n}} - p_{\varepsilon, \delta} \mathbf{n} \right) \cdot \phi ds \right) \\ & \quad + \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla_{x^{(e_l)}, \prime} \mathbf{u}_{\varepsilon, \delta} : \nabla_{x^{(e_l)}, \prime} \phi dx^{(e_l), \prime}, \end{aligned}$$

where for $e_l = \overline{O_i O_j}$, d_l is the distance between the cross sections S_{ij} and S_{ji} , and \mathbf{n} is an outer normal vector for $B_i^{\varepsilon, \delta}$. Using the condition

$$(4.4) \quad \int_{\partial B_i^{\varepsilon, \delta}} \phi \cdot \mathbf{n} = 0, i = 1, \dots, N,$$

we will prove in the Appendix A.3 that the sum of the last two sums of integrals is equal to zero.

So, the variational formulation is as follows: find the vector-field $\mathbf{u}_{\varepsilon, \delta}$ and the pressure $p_{\varepsilon, \delta}$ such that $\mathbf{u}_{\varepsilon, \delta}(x, 0) = 0$, $\mathbf{u}_{\varepsilon, \delta} \in L^\infty(0, T; \mathbf{H}_{div 0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1, \delta}(B_\varepsilon))$, $\mathbf{u}_{\varepsilon, \delta, t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon))$, $\mathbf{u}_{\varepsilon, \delta} = \mathbf{g}_\varepsilon$ on γ_ε , $p_{\varepsilon, \delta} \in L^2(0, T; L^2(B_i^{\varepsilon, \delta}))$ for all $i = 1, \dots, N$, and the couple $(\mathbf{u}_{\varepsilon, \delta}, p_{\varepsilon, \delta})$ satisfies the integral identity for every vector-field $\phi \in \mathbf{H}_0^{1, \delta}(B_\varepsilon)$ for all $t \in (0, T)$,

$$(4.5) \quad \int_{B_\varepsilon} (\mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon, \delta} : \nabla \phi + (\mathbf{u}_{\varepsilon, \delta}, \nabla \mathbf{u}_{\varepsilon, \delta}) \cdot \phi) dx = \sum_{i=1}^N \int_{B_i^{\varepsilon, \delta}} p_{\varepsilon, \delta} \operatorname{div} \phi dx.$$

Note that the so-defined pressure is not unique; it is defined up to function $\theta_i(t)$ in each subdomain $B_i^{\varepsilon, \delta}$, $i = 1, \dots, N$.

A similar weak formulation with pressure can be given for the Stokes equations.

Note that if $N = M + 1$ (number of nodes and vertices is equal to the number of edges plus one), then the restriction (4.4) can be removed from the definition of space

$\mathbf{H}_0^{1,\delta}(B_\varepsilon)$ and then the number of undetermined constants $\theta_i(t)$ in the variational formulation (4.5) will be reduced to one function $\theta_N(t)$, so that the pressure in the reduced geometry is defined up to a constant as in the case of full geometry. This assertion will be proved in the Appendix A.3.

The numerical tests are held for such geometries with $N = M + 1$. In this case it is possible to apply the restriction $\operatorname{div} \mathbf{u}_{\varepsilon,\delta} = 0$ on the solution directly in (4.5) so that a considerably simpler to implement formulation holds true: find the vector-field $\mathbf{u}_{\varepsilon,\delta}$ and the pressure $p_{\varepsilon,\delta}$ such that $\mathbf{u}_{\varepsilon,\delta}(x, 0) = 0$, $\mathbf{u}_{\varepsilon,\delta} \in L^\infty(0, T; \mathbf{H}^1(B_i^{\varepsilon,\delta}))$, for all $i = 1, \dots, N$, $\mathbf{u}_{\varepsilon,\delta,t} \in L^2(0, T; \mathbf{L}^2(B_i^{\varepsilon,\delta}))$, $\mathbf{u}_{\varepsilon,\delta} = \mathbf{g}_\varepsilon$ at γ_ε , $\mathbf{u}_{\varepsilon,\delta} = \mathbf{0}$ at $(\partial B_i^{\varepsilon,\delta} \cap \partial B_\varepsilon) \setminus \gamma_\varepsilon$, $p_{\varepsilon,\delta} \in L^2(0, T; L^2(B_i^{\varepsilon,\delta}))$ for all $i = 1, \dots, N$, $\mathbf{u}_{\varepsilon,\delta} \cdot \mathbf{t} = 0$ on $S_{ij} \cup S_{ji}$, $\mathbf{u}_{\varepsilon,\delta} \cdot \mathbf{n}|_{S_{ij}} + \mathbf{u}_{\varepsilon,\delta} \cdot \mathbf{n}|_{S_{ji}} = 0$, where \mathbf{t} is the unit tangent vector, and the couple $(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta})$ satisfies for all $t \in (0, T)$ the integral identity for every vector-field $\phi \in \mathbf{H}^1(B_i^{\varepsilon,\delta})$, $q \in L^2(B_i^{\varepsilon,\delta})$, for all $i = 1, \dots, N$, such that $\phi = 0$ at $\partial B_i^{\varepsilon,\delta} \cap \partial B_\varepsilon$, and for all edges $O_i O_j$, $\phi \cdot \mathbf{t} = 0$ at $S_{ij} \cup S_{ji}$, and $\phi \cdot \mathbf{n}|_{S_{ij}} + \phi \cdot \mathbf{n}|_{S_{ji}} = 0$:

$$\begin{aligned} \sum_{i=1}^N \int_{B_i^{\varepsilon,\delta}} \left(\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon,\delta} : \nabla \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi - p_{\varepsilon,\delta} \operatorname{div} \phi + q \operatorname{div} \mathbf{u}_{\varepsilon,\delta} \right) dx \\ + \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon,\delta,t} \cdot \phi + \nu \nabla_{x^{(e_l)},'} \mathbf{u}_{\varepsilon,\delta} : \nabla_{x^{(e_l)},'} \phi dx^{(e_l),'} = 0. \end{aligned}$$

Finally, note that the last two terms in (4.4) are analogous to the ones obtained in the context of the so-called Stokes-consistent methods for backflow stabilization at open boundaries [4].

4.5. Estimate for the difference between the exact solution and the MAPDD solution for the nonstationary Navier–Stokes equations. The result of the previous section can be generalized for the nonstationary problem for the Navier–Stokes equations (2.1) using the approach of [15] and [16].

THEOREM 4.3. *Let $\mathbf{g}_j \in C^{[\frac{J+4}{2}]+1}([0, T]; W^{3/2,2}(\partial\omega_j))$. Given natural number J there exists a constant C (independent of ε and J) such that if $\delta = CJ\varepsilon |\ln \varepsilon|$, then*

$$(4.6) \quad \sup_{t \in (0, T)} \|\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} + \|\nabla(\mathbf{u}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon)\|_{L^2((0, T); \mathbf{L}^2(B_\varepsilon))} = O(\varepsilon^J) \quad .$$

5. Numerical examples. In this section, the previous analysis is complemented by numerical experiments for the new MAPDD formulation applied to the stationary Stokes problem and the transient Navier–Stokes problem, for a sequence of values of ε . In the tests we used a more natural Neumann’s condition for the outflow. The errors of the MAPDD solutions obtained in the truncated domain with respect to reference solutions computed in the full domain are evaluated in the norms given by (4.2), (4.6).

5.1. Problem setup. Consider the two-dimensional geometry illustrated in Figure 2.1. Two junctions are connected by a straight tube. This straight tube (labeled $B_{1,2}^{dec,\varepsilon}$) is included in the full reference model or is truncated when the reduced MAPDD model is used.

The radius of the tube is proportional to ε (we set $R = \varepsilon$). For each value of ε , the junction domains are contracted homothetically by a factor of ε with respect to the center points marked with plus signs in Figure 2.1. The distance between

these points, L , remains the same for all values of ε . Straight tube extensions (blue areas, $B_{1;2}^{\varepsilon,\delta}$) are added to the junction domains. Theorem 4.2 requires the associated distance, δ , from the centers of the junction domains to the interfaces, to be

$$(5.1) \quad \delta = C\varepsilon |\ln(\varepsilon)|.$$

C is a user parameter. Pairs of full and reduced domains are created for a sequence of values $\varepsilon = 2^{-k}$, $k = 1, \dots, 6$. In the particular examples of the investigated geometry and our selection of ε , $1/\ln(2) < C < 6/\ln(2)$ is necessary for $B_{1;2}^{\varepsilon,\delta} \neq \emptyset$ and for $B_{1;2}^{dec,\varepsilon} \neq \emptyset$, respectively. In what follows, we choose the values $C = K/\ln(2)$, $K = 2, 3$, and 4 . The factor $1/\ln(2)$ is added for convenience, to cancel the $\ln(\varepsilon)$ terms and leave rational numbers as the interface coordinates.

5.2. Stationary Stokes test case. Since one of our main motivations is the numerical simulation of blood flows, we choose the viscosity and the density values that represent physiologically relevant conditions, assuming the fluid is incompressible and Newtonian. Typical parameters of blood are a dynamic viscosity of $\mu = 0.035 \text{ cm}^2/\text{s}$ and a density of $\rho = 1 \text{ g/cm}^3$. Recall the relation between the dynamic viscosity μ and the kinematic viscosity ν : $\nu = \mu/\rho$. At the inlet Γ_{in} of the upstream junction domain a Dirichlet boundary condition for the velocity is defined as $\mathbf{g}_\varepsilon = (0, 1.5U_0(1 - (x_1 - c_0)^2/\varepsilon^2))^T$, where c_0 is the x_1 coordinate of the center of the boundary and U_0 is chosen such that $Re = 2\rho\varepsilon U_0/\mu = 1$. A homogeneous Neumann boundary condition for the normal stress is applied on the outlet Γ_{out} of the downstream junction domain.

5.3. Transient Navier–Stokes test case. In the transient Navier–Stokes test case, the physical constants are set to the same values as for the Stokes problem, i.e., $\mu = 0.035 \text{ cm}^2/\text{s}$ and $\rho = 1 \text{ g/cm}^3$. A pulsating inflow velocity is defined on Γ_{in} via Dirichlet boundary conditions as $\mathbf{g}_\varepsilon = (0, 1.5U_0(1 - (x_1 - c_0)^2/\varepsilon^2)\sin(\pi t/T))^T$, where t is the actual time and $T = 0.8 \text{ s}$ is the duration of a cycle. U_0 is computed from the Reynolds number, $Re = 2\rho\varepsilon U_0/\mu$. As for the Stokes problem, a homogeneous Neumann boundary condition defines the outflow on Γ_{out} . For the convergence study, Reynolds numbers $Re = 1, 25, 50, 80$, and 100 are considered. In addition, we analyze the MAPDD model for a high Reynolds number of $Re = 2500$.

5.4. Numerical discretization. A mixed finite element method is adopted for discretizing the Stokes and Navier–Stokes equations. We use monolithic velocity–pressure coupling with inf-sup stable second order Taylor–Hood elements on unstructured, uniform triangle meshes. The transient Navier–Stokes problem is discretized in time with the implicit Euler method. The convection term, written in skew-symmetric form, is treated semi-implicitly. The time step size is $\Delta t = 0.01 \text{ s}$. The time interval of the simulations is a half cycle, i.e., $0 \leq t \leq T/2$. The numerical meshes of the domains are created such that the number of elements along the tube diameter is approximately 20 for each value of ε . The average grid size at the boundaries is therefore $h = \varepsilon/10$. This results in 170592 elements in the full domain for the smallest value of $\varepsilon = 2^{-6}$ and $C = 2/\ln(2)$, which corresponds to 784037 degrees of freedom in the Navier–Stokes system. The triangulation of the corresponding reduced domain consists of 15366 elements and the solution space contains 70741 degrees of freedom. The problem is implemented and solved using the FEniCS finite element library [1]. The numerical meshes are generated with Gmsh [9].

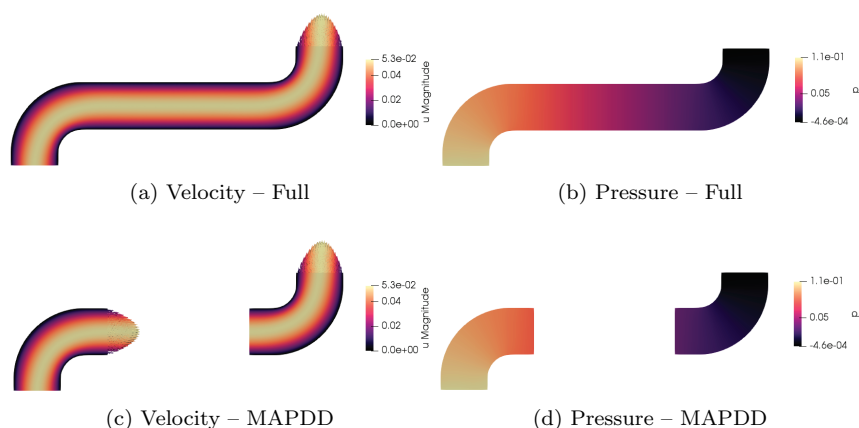


FIG. 5.1. Pressure fields and velocity magnitude and vectors at the outflow boundaries obtained for the stationary Stokes problem using $\varepsilon = 0.5$ with the full model (top row) and with the MAPDD model (bottom row).

5.5. Results.

5.5.1. Stationary Stokes test case. The velocity and pressure field of the stationary Stokes problem, computed with the full model and with the MAPDD method, are illustrated in Figure 5.1 for the largest value of $\varepsilon = 0.5$. No visible differences exist between the full and the MAPDD results.

The velocity error of the MAPDD model with respect to the full reference solution is analyzed quantitatively in Figure 5.2 for the full range of values of ε . The error is computed in the $\mathbf{H}^1(B_\varepsilon)$ norm; cf. (4.2) in Theorem 4.2. Note that the error estimate depends on the solutions in the full domain, B_ε . The mesh nodes of the MAPDD and the full domains match for the junctions. In the truncated tube, the MAPDD solution was interpolated from the interfaces, $\Sigma_{1,2}$, to the mesh nodes of the full mesh. The rate of convergence can be estimated from the numerical results as

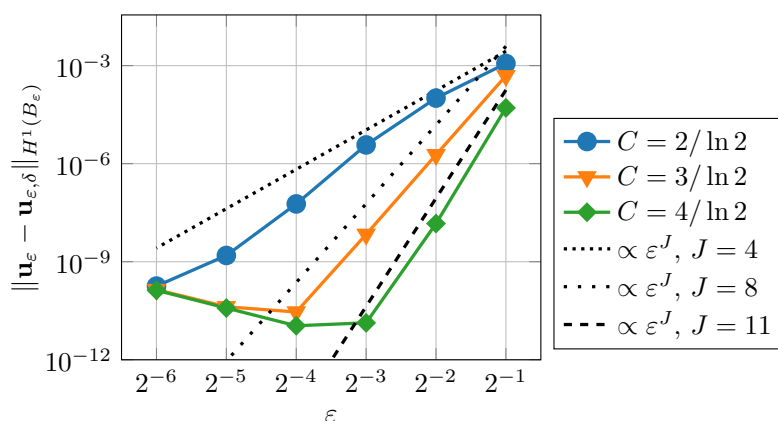


FIG. 5.2. Stationary Stokes test case: convergence of the error with respect to ε for different values of C (see legend).

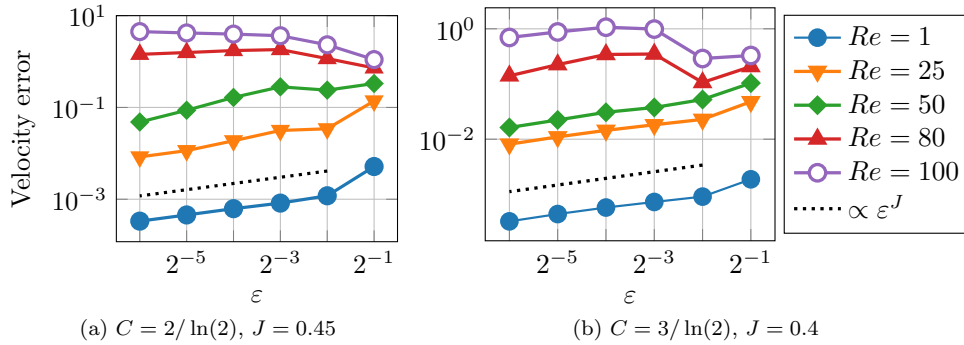


FIG. 5.3. Errors (equation (4.6)) of the Navier-Stokes MAPDD model w.r.t. to the full solution for different Reynolds numbers for different values of C .

$$J_k = \frac{\log e_k / e_{k-1}}{\log \varepsilon_k / \varepsilon_{k-1}},$$

where $e_k = \|\mathbf{u}_{\varepsilon_k} - \mathbf{u}_{\varepsilon_k, \delta}\|_{\mathbf{H}^1(B_{\varepsilon_k})}$, $\varepsilon_k = 2^{-k}$, $k = 2, \dots, 6$. While not constant, for $C = 2/\ln(2)$, J_k is in the range $3 \lesssim J_k \lesssim 6$. The error drops at least with cubic convergence (in the investigated cases). For $C = 3/\ln(2)$ the convergence rate is greatly improved, and even more so using $C = 4/\ln(2)$, namely, we obtain $J \approx 8$ and $J \approx 11$, respectively, discarding the points where the error stagnates. The stagnation of both cases for $\varepsilon < 2^{-4}$ or 2^{-3} is due to the precision of the numerical method being reached. Rounding errors gain importance for very small values of ε .

5.5.2. Transient Navier-Stokes test case. The asymptotic behavior of the error of the MAPDD method with respect to the full model is shown for different Reynolds numbers in Figure 5.3(a) for $C = 2/\ln(2)$. The error is evaluated in the norm (4.6). For the lowest investigated Reynolds number $Re = 1$, the rate of convergence J was computed (omitting the two largest values of ε). The line ε^J is included in the figure for comparison. With increasing Reynolds numbers the rate of convergence decreases. Exponential increase of the error was observed for $Re = 100$. Using $C = 3/\ln(2)$ (see Figure 5.3(b)), the rate of convergence obtained for Reynolds numbers $Re > 1$ is improved. In particular, for $Re = 100$ the error now decreases with ε , at least for small values of ε . The errors of the case $Re = 100$ obtained for $C = K/\ln(2)$, $K = 2, 3, 4$, are shown in Figure 5.4. Indeed, for higher K , the errors are lower and convergence is improved for $\varepsilon \lesssim 2^{-4}$. While the error estimate assumes a low Reynolds number, the MAPDD method can still be applied to these cases. Figures 5.5 and 5.6 show velocity streamlines and the pressure field obtained with the full reference model and the MAPDD method applied to the case $\varepsilon = 1/4$ and for a Reynolds number of $Re = 2500$, as an example. The boundary mesh size was set to $h = \varepsilon/20$; furthermore $C = 2/\ln(2)$. The results match very well visually. The MAPDD model is able to recover the recirculation zones in both junctions accurately (Figures 5.5(a) and (b)). For a more detailed comparison, the axial velocity profiles at the interfaces for the MAPDD solution and for the full solution in the corresponding location are shown in Figure 5.7. At the left interface, the velocity interface conditions produce a pressure overshoot near the upper corner, since the Womersley hypothesis is in disagreement with the high Reynolds number flow conditions. This can be seen more clearly in Figure 5.8(a), where the pressure profile at the interface is shown for both the MAPDD and the full solution. However, analyzing the pressure distribution along

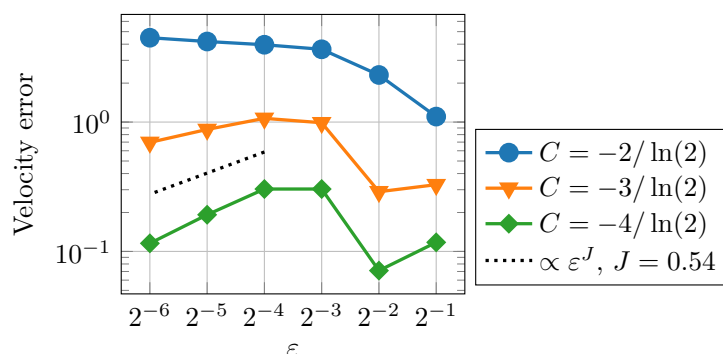


FIG. 5.4. Comparison of the Navier–Stokes error with different values of C for $Re = 100$.

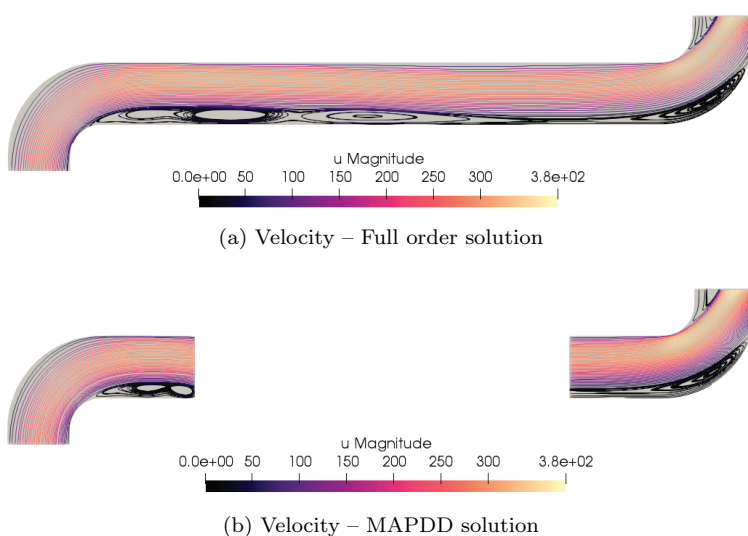


FIG. 5.5. Velocity stream lines of the transient Navier–Stokes test case at peak time $t = 0.2$ s for $Re = 2500$, $\varepsilon = 0.25$. Full model (a) versus MAPDD model (b).

the cross section the tube in a slightly more upstream position (shifted upstream by 2ε), the MAPDD recovers the behavior observed for the full solution with an error of $< 8\%$ (Figure 5.9). The pressure on the right interface does not suffer any nonphysical oscillations, as can be seen in Figure 5.8(b), and the discrepancy between both models is within 2%.

5.6. Conclusion. The MAPDD was shown to be an efficient and accurate method for the steady Stokes problem and for the low Reynolds number Navier–Stokes problem. In these cases, the error of the MAPDD method was in agreement with theoretical error estimates, (4.2) and (4.6), respectively. For slightly larger Reynolds numbers, the convergence can be improved by modifying the computational domain and adjusting the constant in (5.1). Although the theory is valid only for small Reynolds numbers, the method yields very good results also for high Reynolds numbers. For the (arbitrary) example of $Re = 2500$, $\varepsilon = 1/4$, the MAPDD velocity and pressure solutions were in good agreement with the full solution, except for pressure oscillations that occur near the upstream interface.

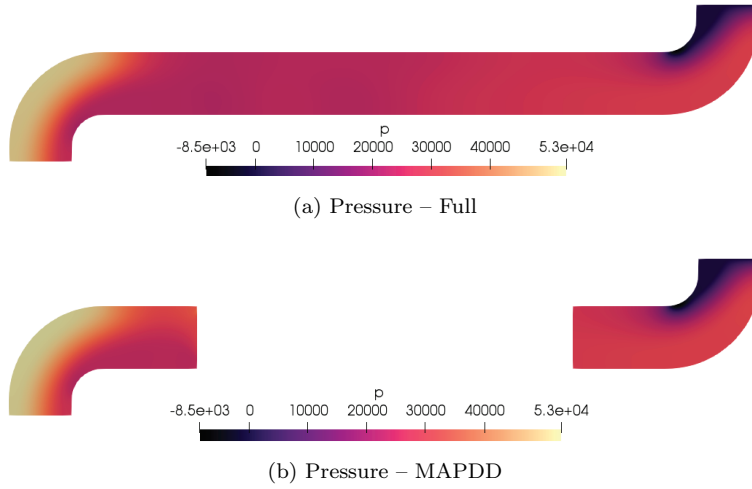


FIG. 5.6. Pressure fields of the transient Navier-Stokes test case at peak time $t = 0.2$ s for $Re = 2500$, $\varepsilon = 0.25$. Full model (a) versus MAPDD model (b).

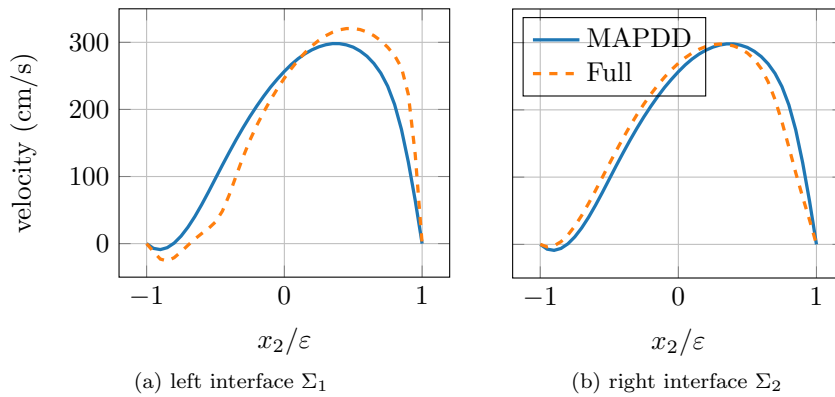


FIG. 5.7. Axial velocity component u_0 at the interfaces for the MAPDD and the full solutions computed for $Re = 2500$, $\varepsilon = 1/4$.

Appendix A. Proofs of the main theorems. Consider the steady state Stokes equations (2.6). Let us give a weak formulation form equivalent to Definitions 1.1' and 1.2' introducing a new unknown function $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{g}_\varepsilon$, which is divergence free and vanishing at the whole boundary.

DEFINITION 3'. By a weak solution we understand the vector $\mathbf{u}_\varepsilon \in \mathbf{H}_{div0}^1(\partial B_\varepsilon \setminus \gamma_\varepsilon)$ (B_ε) such that the difference $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{g}_\varepsilon$ belongs to $\mathbf{H}_{div0}^1(B_\varepsilon)$ and satisfies the following integral identity: for all $\mathbf{v} \in \mathbf{H}_{div0}^1(B_\varepsilon)$,

$$(A.1) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{v}_\varepsilon(x) : \nabla \mathbf{v}(x) dx = -\nu \int_{B_\varepsilon} \nabla \mathbf{g}_\varepsilon(x) : \nabla \mathbf{v}(x) dx.$$

It is well known that there exists a unique solution to this problem (see [11]). Further we will need as well a modification of this problem containing a right-hand-side $\mathbf{f}_\varepsilon = \mathbf{f}_{0\varepsilon} - \sum_{i=1}^n \frac{\partial \mathbf{f}_{i\varepsilon}}{\partial x_i}$, where $\mathbf{f}_{i\varepsilon} \in \mathbf{L}^2(B_\varepsilon)$, $i = 0, 1, \dots, n$,

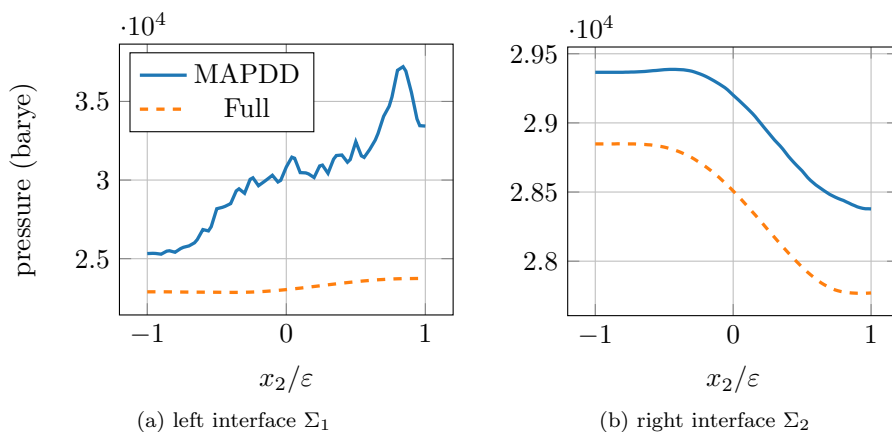


FIG. 5.8. Pressure along the interfaces for the MAPDD and the full solutions computed for $Re = 2500$, $\varepsilon = 1/4$.

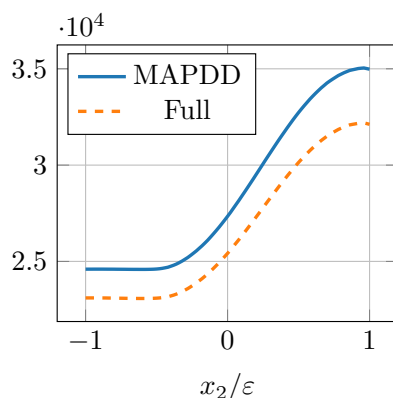


FIG. 5.9. Pressure along the tube cross section, at 2ε upstream of Σ_1 , for the MAPDD and the full solutions computed for $Re = 2500$, $\varepsilon = 1/4$.

$$(A.2) \quad \begin{aligned} -\nu \Delta \mathbf{v}_\varepsilon + \nabla p_\varepsilon &= \mathbf{f}_\varepsilon, \quad \operatorname{div} \mathbf{v}_\varepsilon = 0, \quad x \in B_\varepsilon, \\ \mathbf{v}_\varepsilon &= 0, \quad x \in \partial B_\varepsilon, \end{aligned}$$

with a variational formulation: to find a vector-valued function $\mathbf{v}_\varepsilon \in \mathbf{H}_{div0}^1(B_\varepsilon)$ such that for any test function $\mathbf{v} \in \mathbf{H}_{div0}^1(B_\varepsilon)$ it satisfies the following integral identity: for all $\mathbf{v} \in \mathbf{H}_{div0}^1(B_\varepsilon)$,

$$(A.3) \quad \nu \int_{B_\varepsilon} \nabla \mathbf{v}_\varepsilon(x) : \nabla \mathbf{v}(x) dx = \int_{B_\varepsilon} \mathbf{f}_{0\varepsilon} \cdot \mathbf{v}(x) + \sum_{i=1}^n \mathbf{f}_{i\varepsilon} \cdot \frac{\partial \mathbf{v}}{\partial x_i}(x) dx.$$

This problem as well admits a unique solution and using the well-known Poincaré–Friedrichs inequality with a constant proportional to ε (see, for example, [20]) we get an a priori estimate:

$$(A.4) \quad \|\mathbf{v}_\varepsilon\|_{\mathbf{H}^1(B_\varepsilon)} \leq \bar{C} \sum_{i=0}^n \|\mathbf{f}_{i\varepsilon}\|_{\mathbf{L}^2(B_\varepsilon)},$$

where \bar{C} does not depend on \mathbf{f}_ε .

A.1. Divergence-free extension of boundary value function having the steady Poiseuille shape in the cylinders. Let us prove that there exists a special divergence-free extension \mathbf{g}_ε such that within some interior part of the cylinders $B_\varepsilon^{(e)}$ it is a Poiseuille flow (3.1).

LEMMA A.1. *Function \mathbf{g}_ε can be extended to B_ε so that its divergence free extension $\mathbf{g}_\varepsilon \in H^1(B_\varepsilon)$, for each subdomain $\tilde{\omega}_\varepsilon^j = \omega_\varepsilon^j \cup (B_\varepsilon^{(e)} \cap \{x_1^{(e)} < \varepsilon d\})$ (here e is the edge containing $O_j, j = N_1 + 1, \dots, N$) on γ_ε^j it is equal to $\mathbf{g}_\varepsilon(x) = \mathbf{g}_j(\frac{x - O_j}{\varepsilon})$, and for any cylinder $B_\varepsilon^{(e)} \cap \{3\varepsilon d < x_1^{(e)} < |e| - 3\varepsilon d\}$ is equal in local coordinates to a Poiseuille flow $\mathbf{V}_P(x)$.*

Proof. Let us consider a problem for the pressure p on the graph \mathcal{B} (in this proof we omit for simplicity the index ε) (see [15]) with the continuity condition for the pressure on the graph and with given derivatives of the pressure at the vertices (corresponding to the flow rates $\int_{\gamma_\varepsilon^j} \mathbf{g}_\varepsilon \cdot \mathbf{n} ds$):

$$\begin{aligned} \kappa_j \frac{\partial^2 p}{\partial x_1^{(e_j)2}} &= 0, x \in e_j, j = 1, \dots, M, \\ \sum_{j: e_j \in O_i} \kappa_j \frac{\partial p}{\partial x_1^{(e_j)}} &= 0, i = 1, \dots, N_1, \\ \kappa_j \frac{\partial p}{\partial x_1^{(e_j)}} &= \int_{\gamma_\varepsilon^j} \mathbf{g}_\varepsilon \cdot \mathbf{n} ds, i = N_1 + 1, \dots, N, \\ p &\text{ is continuous function on the graph } \mathcal{B}. \end{aligned}$$

Here $\kappa_j = \varepsilon^{n-1} \int_{\sigma_j} v_P(y') dy'$, and the local axes have the origin O_i . This problem admits a unique (up to an additive constant) solution, linear function on every edge; see [14]. The slope of this solution at each edge defines the Poiseuille flow in the corresponding cylinder: $\mathbf{V}_P^{(e_j)}$, such that

$$\kappa_j \frac{\partial p}{\partial x_1^{(e_j)}} = \int_{\sigma_{j\varepsilon}} \mathbf{V}_P^{(e_j)} \cdot \mathbf{n} ds,$$

$\sigma_{j\varepsilon} = \{x \in B_\varepsilon^j | x_1 = 0\}$. Then for every domain $\tilde{\omega}_\varepsilon^j$ we construct a divergence-free extension equal to the determined Poiseuille flows on the parts of the boundary coinciding with the sections of the cylinders. The possibility of this construction is ensured by the flux balance in every node of the graph (see [12], [22], [21], [7]). \square

LEMMA A.2. *The extension may be constructed in such a way that it belongs to $\mathbf{H}^2(B_\varepsilon)$.*

Proof. Indeed, for every edge e_i multiply constructed Poiseuille flow $\mathbf{V}_P^{(e_i)}$ by a cut-off function $\zeta(\frac{x_1^{(e_i)}}{3d\varepsilon})\zeta(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon})$, where $\zeta(\tau)$ is a smooth cut-off function independent of ε with $\zeta(\tau) = 0$ for $\tau \leq 1/3$ and $\zeta(\tau) = 1$ for $\tau \geq 2/3$, $0 \leq \zeta(\tau) \leq 1$. Denote $\psi_\varepsilon = \operatorname{div}(\zeta(\frac{x_1^{(e_i)}}{3d\varepsilon})\zeta(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon})\mathbf{V}_P^{(e_i)}(x))$. Evidently, $\psi_\varepsilon \in \mathbf{H}_0^1(B_\varepsilon)$ and $\int_{B_\varepsilon} \psi_\varepsilon dx = 0$. Then for every domain $\tilde{\omega}_\varepsilon^j$ we can construct a function \mathbf{w}_ε such that $\operatorname{div} \mathbf{w}_\varepsilon = -\psi_\varepsilon$ and $\mathbf{w}_\varepsilon \in \mathbf{H}_0^2(\tilde{\omega}_\varepsilon^j), j = 1, \dots, N_1$ (see [7], [10]), and $\mathbf{w}_\varepsilon \in \mathbf{H}_0^2(\tilde{\omega}_\varepsilon^j), j = N_1 + 1, \dots, N$, moreover, \mathbf{w}_ε and its gradient vanish on the part of $\partial\tilde{\omega}_\varepsilon^j$ belonging to B_ε , and $\mathbf{w}_\varepsilon = \mathbf{g}_\varepsilon$ on γ_ε^j . We take $\tilde{\mathbf{g}}_\varepsilon = \mathbf{w}_\varepsilon + \zeta(\frac{x_1^{(e_i)}}{3d\varepsilon})\zeta(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon})\mathbf{V}_P^{(e_i)}(x)$. \square

A.2. Estimate for the difference between the exact solution and the MAPDD solution: Asymptotic analysis of the Stokes equations.

Proof of Theorem 4.2. 1. Consider an asymptotic expansion of the solution (see [18], [6]). For the velocity \mathbf{u}_ε it has the following shape: a Poiseuille flow $\mathbf{V}_P(x)$ within the cylinders $B_\varepsilon^{(e)}$ plus some functions depending on the variable $\frac{x-O_l}{\varepsilon}$ exponentially tending to zero in the cylinders as the variable $\frac{x-O_l}{\varepsilon}$ tends to infinity. One can write this expansion as a uniform approximation of order J in a form

$$(A.5) \quad \mathbf{v}_\varepsilon^{(J)}(x) = \sum_{i=1}^M \zeta\left(\frac{x_1^{(e_i)}}{3d\varepsilon}\right) \zeta\left(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon}\right) \mathbf{V}_P^{[i,J]}\left(\frac{x^{(e)'}}{\varepsilon}\right) + \sum_{l=1}^N \left(1 - \zeta\left(\frac{|x - O_l|}{|e|_{\min}}\right)\right) \mathbf{V}^{[BLO_l, J]}\left(\frac{x - O_l}{\varepsilon}\right).$$

Here $\mathbf{V}_P^{[i,J]}(y^{(e)'}) = C^{[i,J]}(k_{e1}v_P(y^{(e)'}), \dots, k_{en}v_P(y^{(e)'}))^T$ (as above, e has the cosines directors k_{e1}, \dots, k_{en} and the local variables $x^{(e)}$ are related to the global ones by the equation $x^{(e)} = x^{(e)}(x)$), $C^{[i,J]}$ are constants such that for any node O_l the flux conservation law is satisfied,

$$(A.6) \quad \sum_{i: O_l \in e_i} C^{[i,J]} \int_{\sigma^{(e_i)}} v_P(y^{(e)'}) dy^{(e)'} = 0$$

(the local coordinate system $x^{(e)}$ has its origin in O_l), and for vertices O_l , the end points of only one edge e_i ,

$$(A.7) \quad C^{[i,J]} \int_{\sigma^{(e_i)}} v_P(y^{(e)'}) dy^{(e)'} + \int_{\gamma_l^\varepsilon} \mathbf{g}_\varepsilon \cdot \mathbf{n} ds \varepsilon^{1-n} = 0$$

(\mathbf{n} is an outer normal vector). Every $\mathbf{V}_P^{[i,J]}(y^{(e)'})$ is defined only within the cylinder $B_\varepsilon^{(e)}$ associated to the edge e . Here $|e|_{\min}$ is the minimal length of the edges. The boundary layer term, vector-valued function $\mathbf{V}^{[BLO_l, J]}(y)$, exponentially vanishes as $|y|$ tends to infinity: in the sense that the product $\mathbf{V}^{[BLO_l, J]}(y)e^{\beta|y|}$ belongs to the space $\mathbf{H}^1(\Omega_l)$ with some positive β independent of ε , $\Omega_l = \{y \in \mathbb{R}^n | \varepsilon(y + O_l) \in \tilde{B}_l^{\varepsilon, \delta}\}$, and $\tilde{B}_l^{\varepsilon, \delta}$ is an extension of $B_l^{\varepsilon, \delta}$ behind the cross sections S_{ij} as semi-infinite cylinders: $\tilde{B}_l^{\varepsilon, \delta} = B_l^{\varepsilon, \delta} \cup_{e: O_l \in e} \{x_1^{(e)} > \varepsilon d; x^{(e)'}/\varepsilon \in \sigma^{(e)}\}$. So, Ω_l is an unbounded domain obtained from the bounded domain $\{\varepsilon(y + O_l) \in B_l^{\varepsilon, \delta}\}$ with truncated cylinders by extension of them behind the truncations, so that they become the cylindrical outlets to infinity. Functions $\mathbf{V}^{[BLO_l, J]}$ satisfy the integral identity,

$$(A.8) \quad \int_{\Omega_l} \nu \nabla \mathbf{V}^{[BLO_l, J]}(y) : \nabla \Phi(y) dy + \sum_{i: O_l \in e_i} \int_{\Omega_l} \nu \nabla \left(\mathbf{V}_P^{[i,J]}(y^{(e)'}) \zeta\left(\frac{y_1^{(e_i)}}{3d}\right) \right) : \nabla \Phi(y) dy = 0$$

for all divergence-free vector-valued test functions Φ from $\mathbf{H}_0^1(\Omega_l)$ and

$$\operatorname{div} \left(\mathbf{V}^{[BLO_l, J]}(y) + \sum_{i: O_l \in e_i} \left(\mathbf{V}_P^{[i,J]}(y^{(e)'}) \zeta\left(\frac{y_1^{(e_i)}}{3d}\right) \right) \right) = 0.$$

$\mathbf{V}^{[BLO_l, J]}$ satisfy the homogeneous Dirichlet boundary conditions if O_l is an end point for at least two edges,

$$(A.9) \quad \mathbf{V}^{[BLO_l, J]}(y) = 0, \quad y \in \partial\Omega_l,$$

or the nonhomogeneous Dirichlet boundary conditions

$$(A.10) \quad \mathbf{V}^{[BLO_l, J]}(y) = \mathbf{g}_l(y), \quad y \in \gamma_l; \quad \mathbf{V}^{[BLO_l, J]}(y) = 0, \quad y \in \partial\Omega_l \setminus \gamma_l;$$

in the case if O_l is a vertex, $\gamma_l = \partial\Omega_l \cap \partial\omega_l$.

For the asymptotic expansion the estimate (see [20], [6])

$$(A.11) \quad \|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon^{(J)}\|_{H^1(B_\varepsilon)} = O(\varepsilon^J)$$

holds.

2. Let us multiply all boundary layers by a cut-off function passing from value one within the distance less than $\delta/3$ from the nodes to the value zero if the distance from the nodes is greater than $2\delta/3$, i.e., we replace $\mathbf{v}_\varepsilon^{(J)}(x)$ by the new asymptotic approximation

$$(A.12) \quad \begin{aligned} \mathbf{u}_\varepsilon^a(x) = & \sum_{i=1}^M \zeta\left(\frac{x_1^{(e_i)}}{3d\varepsilon}\right) \zeta\left(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon}\right) \mathbf{V}_P^{[i, J]}(x^{(e)'}/\varepsilon) \\ & + \sum_{l=1}^N \left(1 - \zeta\left(\frac{|x - O_l|}{\delta}\right)\right) \mathbf{V}^{[BLO_l, J]}\left(\frac{x - O_l}{\varepsilon}\right). \end{aligned}$$

This new approximation consists only of the Poiseuille flow within the cylinders $B_\varepsilon^{(e)}$ at the distance δ from the ends of e because the cut-off function $1 - \zeta(\frac{|x - O_l|}{\delta})$ vanishes in this area. Let us choose δ equal $C_J \varepsilon |\ln \varepsilon| |e|_{\min}$ and choose the constant C_J such that the residual in the right-hand side of the equations has the order $O(\varepsilon^J)$. To this end notice that the boundary layer functions decay exponentially, i.e., $\mathbf{V}^{[BLO_l, J]}(y) e^{\beta|y|}$ belongs to the space $\mathbf{H}^1(\Omega_l)$ with some positive β independent of ε . Let us find δ such that

$$(A.13) \quad \mathcal{F}_{l, \frac{\delta}{\varepsilon}} = O(\varepsilon^{J+2}), \quad \mathcal{F}_{l, R} = \|\mathbf{V}^{[BLO_l, J]}\|_{\mathbf{H}^1(\Omega_{l, R})},$$

where $\Omega_{l, R} = \Omega_l \cap \{|y| > R\}$. Indeed, the inclusion $\mathbf{V}^{[BLO_l, J]}(y) e^{\beta|y|} \in H^1(\Omega_l)$ implies

$$e^{\beta \frac{\delta}{\varepsilon}} \mathcal{F}_{l, \frac{\delta}{\varepsilon}} \leq \|e^{\beta|y|} \mathbf{V}^{[BLO_l, J]}(y)\|_{\mathbf{H}^1(\Omega_{l, \frac{\delta}{\varepsilon}})} \leq \|e^{\beta|y|} \mathbf{V}^{[BLO_l, J]}(y)\|_{\mathbf{H}^1(\Omega_l)}$$

and the last norm is bounded by a constant, denote it C_l . So, we can write $e^{\beta \frac{\delta}{\varepsilon}} \mathcal{F}_{l, \frac{\delta}{\varepsilon}} \leq C_l$, i.e., $\mathcal{F}_{l, \frac{\delta}{\varepsilon}} \leq C_l e^{-\beta \frac{\delta}{\varepsilon}}$. Let us take $\delta = \frac{1}{\beta}(J+2)\varepsilon |\ln \varepsilon|$; then we get the estimate (A.13), and so, making the change of variables $x - O_l = \varepsilon y$, we get that the difference $\mathbf{v}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon^a$ has support belonging to the cylinders $B_\varepsilon^{(e), \delta/3} = B_\varepsilon^{(e)} \cap \{\delta/3 < x_1^{(e)} < |e| - \delta/3\}$ and

$$\|\mathbf{v}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon^a\|_{\mathbf{H}^1(B_\varepsilon^{(e), \delta/3})} = O(\varepsilon^J).$$

Thus,

$$\|\mathbf{v}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon^a\|_{\mathbf{H}^1(B_\varepsilon)} = O(\varepsilon^J).$$

3. Unfortunately, \mathbf{u}_ε^a may be not divergence free within the parts $B_\varepsilon^{(e),\delta/3,-} = B_\varepsilon^{(e),\delta/3} \cap \{x_1^{(e)} < 2\delta/3\}$ and $B_\varepsilon^{(e),\delta/3,+} = B_\varepsilon^{(e),\delta/3} \cap \{x_1^{(e)} > |e| - 2\delta/3\}$ of the cylinders $B_\varepsilon^{(e),\delta/3}$ because the products $(1 - \zeta(\frac{|x-O_l|}{\delta}))\mathbf{V}^{[BLO_l,J]}(\frac{x-O_l}{\varepsilon})$ have the divergence equal to $h_l(x) = -\operatorname{div}(\zeta(\frac{|x-O_l|}{\delta})\mathbf{V}^{[BLO_l,J]}(\frac{x-O_l}{\varepsilon})) = -\delta^{-1}\zeta'(\frac{|x-O_l|}{\delta})\nabla|x-O_l| \cdot \mathbf{V}^{[BLO_l,J]}(\frac{x-O_l}{\varepsilon})$. The vector-field $\mathbf{V}^{[BLO_l,J]}$ belongs to $\mathbf{H}_0^1(\Omega_l)$ and, therefore, the flux of it in every outlet to infinity is equal to zero. Let us show that $\int_{B_\varepsilon^{(e),\delta/3,\pm}} h_l(x)dx = 0$. Indeed, integrating the divergence term by parts, we get

$$\begin{aligned} & \int_{B_\varepsilon^{(e),\delta/3,\pm}} \operatorname{div} \left(\zeta \left(\frac{|x-O_l|}{\delta} \right) \mathbf{V}^{[BLO_l,J]} \left(\frac{x-O_l}{\varepsilon} \right) \right) dx \\ &= \int_{\sigma_\varepsilon^{(e)}(x_1^{(e)}=2\delta/3)} \mathbf{V}^{[BLO_l,J]} \left(\frac{x-O_l}{\varepsilon} \right) \cdot \mathbf{n} dS = 0, \end{aligned}$$

where $\sigma_\varepsilon^{(e)}(x_n^{(e)}) = 2\delta/3$ is the section of $B_\varepsilon^{(e)}$ corresponding to $x_1^{(e)} = 2\delta/3$, and we have used that $\zeta(\frac{|x-O_l|}{\delta}) = 0$ for $\frac{|x-O_l|}{\delta} \leq 1/3$, $\zeta(\frac{|x-O_l|}{\delta}) = 1$ for $\frac{|x-O_l|}{\delta} \geq 2/3$, and that $\mathbf{V}^{[BLO_l,J]}$ is equal to zero on the lateral boundary of $B_\varepsilon^{(e)}$. Thus, we can apply the results from [17] (see Lemma 3.1 and the change of variables from Lemma 3.6 in [17]) and prove that there exists $\mathbf{w} \in \mathbf{H}_0^1(B_\varepsilon^{(e),\delta/3,\pm})$ such that $\operatorname{div} \mathbf{w} = -h_l(x)$ and $\|h_l\|_{L^2(B_\varepsilon^{(e),\delta/3,\pm})} \leq \frac{\varepsilon}{\delta} \|\mathbf{V}^{[BLO_l,J]}\|_{\mathbf{L}^2(B_\varepsilon^{(e),\delta/3,\pm})} = O(\frac{1}{\delta}\varepsilon^{J+2+n/2})$ and so $\|\mathbf{w}\|_{\mathbf{H}^1(B_\varepsilon^{(e),\delta/3,\pm})} = O(\varepsilon^J)$. Extending \mathbf{w} by zero outside the domains $B_\varepsilon^{(e),\delta/3,\pm}$ we get an approximation $\mathbf{u}_\varepsilon^{(J)} = \mathbf{u}_\varepsilon^a + \mathbf{w}$ which belongs to the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$.

4. Evidently the difference $\mathbf{u}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon$ satisfies the homogeneous Dirichlet boundary condition on ∂B_ε and the Stokes equations in B_ε with a residual of order $O(\varepsilon^J)$. There holds the following integral identity:

$$(A.14) \quad \nu \int_{B_\varepsilon} \nabla(\mathbf{u}_\varepsilon^{(J)}(x) - \mathbf{u}_\varepsilon(x)) : \nabla \mathbf{v}(x) dx = -\nu \int_{B_\varepsilon} \nabla \mathbf{r}_\varepsilon(x) : \nabla \mathbf{v}(x) dx$$

for all $\mathbf{v} \in \mathbf{H}_{div0}^1(B_\varepsilon)$, where $\mathbf{r}_\varepsilon(x) = \nabla(\mathbf{w}(x) + (\mathbf{u}_\varepsilon^a(x) - \mathbf{v}_\varepsilon^{(J)}(x)) + (\mathbf{v}_\varepsilon^{(J)}(x) - \mathbf{u}_\varepsilon(x)))$.

Due to the previous estimates the norms $\|\mathbf{w}\|_{\mathbf{H}^1(B_\varepsilon)}$, $\|\mathbf{u}_\varepsilon^a - \mathbf{v}_\varepsilon^{(J)}\|_{\mathbf{H}^1(B_\varepsilon)}$, and $\|\mathbf{v}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon\|_{\mathbf{H}^1(B_\varepsilon)}$ are of order $O(\varepsilon^J)$, and we get $\|\mathbf{r}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} = O(\varepsilon^J)$. So, applying the a priori estimate (A.4), we get the following inequality:

$$(A.15) \quad \|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{\mathbf{H}^1(B_\varepsilon)} = O(\varepsilon^J).$$

5. Consider the projection of problem (2.6) on the subspace $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$. By the Lax–Milgram theorem there exists a unique solution $\mathbf{u}_{\varepsilon,\delta}$ to this projection and the difference $\mathbf{u}_\varepsilon^{(J)} - \mathbf{u}_{\varepsilon,\delta}$ belongs to the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ and satisfies, as before (see (A.14)), variational formulation with a residual of order $O(\varepsilon^J)$: for every vector-field $\mathbf{v} \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$,

$$(A.16) \quad \nu \int_{B_\varepsilon} \nabla(\mathbf{u}_\varepsilon^{(J)}(x) - \mathbf{u}_{\varepsilon,\delta}(x)) : \nabla \mathbf{v}(x) dx = -\nu \int_{B_\varepsilon} \nabla \mathbf{r}_\varepsilon(x) : \nabla \mathbf{v}(x) dx.$$

Now applying an a priori estimate (A.4) we get

$$(A.17) \quad \|\mathbf{u}_\varepsilon^{(J)} - \mathbf{u}_{\varepsilon,\delta}\|_{\mathbf{H}^1(B_\varepsilon)} = O(\varepsilon^J).$$

Estimates (A.15), (A.17) imply (4.2). \square

Remark A.3. Notice that in [18], [6], [20] an asymptotic expansion $p_\varepsilon^{(J)}$ of the pressure p_ε was as well constructed and it was similar to the (A.5) structure: a linear pressure depending on the longitudinal variable for each cylinder $B_\varepsilon^{(e)}$ multiplied by a cut-off function plus the boundary layer terms exponentially decaying as $\frac{|x-O_l|}{\varepsilon}$ tends to infinity. For the linear pressure the second order differential equation on the graph with the Kirchhoff-type junction conditions in the nodes holds. The couple $(\mathbf{v}_\varepsilon^{(J)}, p_\varepsilon^{(J)})$ satisfies (2.6) in a classical sense with a residual of order $O(\varepsilon^J)$ in the $L^2(B_\varepsilon)$ -norm. Moreover, the couple $(\mathbf{u}_\varepsilon^{(J)}, p_\varepsilon^a)$, where p_ε^a is obtained from $p_\varepsilon^{(J)}$ replacing the cut-off factor of boundary layers by $1 - \zeta(\frac{|x-O_l|}{\delta})$, as well satisfies (2.6) in the classical sense with a residual of order $O(\varepsilon^J)$ in the $\mathbf{L}^2(B_\varepsilon)$ -norm:

$$(A.18) \quad -\nu \Delta \mathbf{u}_\varepsilon^{(J)}(x) + \nabla p_\varepsilon^a(x) = \mathbf{r}_\varepsilon(x), \quad \operatorname{div} \mathbf{u}_\varepsilon^{(J)} = 0, \quad x \in B_\varepsilon,$$

where $\|\mathbf{r}_\varepsilon\|_{\mathbf{L}^2(B_\varepsilon)} = O(\varepsilon^J)$. The boundary conditions are satisfied exactly.

A.3. Navier–Stokes equations. Consider the Navier–Stokes equations (2.1). Let us remind a weak formulation from [15] which is equivalent to Definitions 1 and 2. It introduces a new unknown function $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{g}_\varepsilon$, which is divergence free and vanishing at the whole boundary.

DEFINITION 3. By a weak solution we understand the vector-field $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{g}_\varepsilon$, where $\operatorname{div} \mathbf{v}_\varepsilon = 0$, $\mathbf{v}_\varepsilon(x, 0) = 0$, $\mathbf{v}_\varepsilon \in L^\infty(0, T; \mathbf{H}_0^1(B_\varepsilon))$, $\mathbf{v}_{\varepsilon t} \in L^2(0, T; \mathbf{L}^2(B_\varepsilon))$, and for every vector-field $\phi \in \mathbf{H}_0^1(B_\varepsilon)$ and for all $t \in (0, T)$, \mathbf{v}_ε satisfies the integral identity

$$(A.19) \quad \begin{aligned} & \int_{B_\varepsilon} \left(\mathbf{v}_{\varepsilon t} \cdot \phi + \nu \nabla \mathbf{v}_\varepsilon \cdot \nabla \phi - ((\mathbf{v}_\varepsilon + \mathbf{g}_\varepsilon) \cdot \nabla) \phi \cdot \mathbf{v}_\varepsilon - (\mathbf{v}_\varepsilon \cdot \nabla) \phi \cdot \mathbf{g}_\varepsilon \right) dx \\ &= -\nu \int_{B_\varepsilon} \nabla \mathbf{g}_\varepsilon \cdot \nabla \phi dx - \int_{B_\varepsilon} ((\mathbf{g}_\varepsilon \cdot \nabla) \mathbf{g}_\varepsilon + \mathbf{g}_{\varepsilon t}) \cdot \phi dx. \end{aligned}$$

The proof of the existence of a solution for sufficiently small values of ε repeats literally the proof from [15].

The new version of the MAPDD replaces the problem (2.1) by (2.5), where the space $\mathbf{H}_{div 0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1, \delta}(B_\varepsilon)$ is replaced by the newly defined space of divergence-free vector-functions having the Womersley form within cylinders $B_{ij}^{dec, \varepsilon}$.

Let us justify the weak MAPDD formulation with pressure (4.5).

LEMMA A.4. Let $\mathbf{u}_{\varepsilon, \delta}$ be a solution to problem (4.3). For all test functions $\phi \in \mathbf{H}_0^{1, \delta}(B_\varepsilon)$,

$$\begin{aligned} & \sum_{i=1}^N \sum_{j: O_i O_j \in \{e_1, \dots, e_M\}} \int_{\partial B_i^{\varepsilon, \delta} \cap S_{ij}} \left(\nu \frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial \mathbf{n}} - p_{\varepsilon, \delta} \mathbf{n} \right) \cdot \phi ds \\ &+ \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla_{x^{(e_l)}, ' } \mathbf{u}_{\varepsilon, \delta} : \nabla_{x^{(e_l)}, ' } \phi dx^{(e_l), ' } = 0. \end{aligned}$$

Proof. 1. Consider in (4.3) the divergence-free test functions ϕ vanishing in all cylinders $B_{ij}^{dec, \varepsilon}$ except for one of them and two adjacent domains $B_i^{\varepsilon, \delta}$ and $B_j^{\varepsilon, \delta}$. We get

$$\int_{B_i^{\varepsilon, \delta} \cup B_{ij}^{dec, \varepsilon} \cup B_j^{\varepsilon, \delta}} \left(\mathbf{u}_{\varepsilon, \delta, t} \cdot \phi + \nu \nabla \mathbf{u}_{\varepsilon, \delta} : \nabla \phi + (\mathbf{u}_{\varepsilon, \delta}, \nabla \mathbf{u}_{\varepsilon, \delta}) \cdot \phi \right) dx = 0.$$

Integrating by parts and taking into account that in cylinders $B_{ij}^{dec,\varepsilon}$ functions $\mathbf{u}_{\varepsilon,\delta}$ and ϕ are independent of the longitudinal variable, we get

$$\begin{aligned} & \int_{B_i^{\varepsilon,\delta} \cup B_{ij}^{dec,\varepsilon} \cup B_j^{\varepsilon,\delta}} \left(\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi - \nu \Delta \mathbf{u}_{\varepsilon,\delta} \cdot \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \right) dx \\ & + \nu \int_{S_{ij}} \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} \cdot \phi ds + \nu \int_{S_{ji}} \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} \cdot \phi ds = 0. \end{aligned}$$

In $B_i^{\varepsilon,\delta}$ and $B_j^{\varepsilon,\delta}$ function $\mathbf{u}_{\varepsilon,\delta}$ satisfies classical Navier–Stokes equations

$$\mathbf{u}_{\varepsilon,\delta,t} - \nu \Delta \mathbf{u}_{\varepsilon,\delta} + (\mathbf{u}_{\varepsilon,\delta}, \nabla) \mathbf{u}_{\varepsilon,\delta} = -\nabla p_{\varepsilon,\delta,l}, \quad l = i, j.$$

So,

$$\begin{aligned} & \int_{B_i^{\varepsilon,\delta}} (-\nabla p_{\varepsilon,\delta,i}) \cdot \phi dx + \int_{B_{ij}^{dec,\varepsilon}} \left(\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi - \nu \Delta \mathbf{u}_{\varepsilon,\delta} \cdot \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \right) dx \\ & + \int_{B_j^{\varepsilon,\delta}} (-\nabla p_{\varepsilon,\delta,j}) \cdot \phi dx + \nu \int_{S_{ij}} \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} \cdot \phi ds + \nu \int_{S_{ji}} \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} \cdot \phi ds = 0. \end{aligned}$$

Integrate the terms with pressure by parts:

$$\begin{aligned} & \int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta,i} \operatorname{div} \phi dx + \int_{B_{ij}^{dec,\varepsilon}} \left(\mathbf{u}_{\varepsilon,\delta,t} \cdot \phi - \nu \Delta \mathbf{u}_{\varepsilon,\delta} \cdot \phi + (\mathbf{u}_{\varepsilon,\delta}, \nabla \mathbf{u}_{\varepsilon,\delta}) \cdot \phi \right) dx \\ & + \int_{B_j^{\varepsilon,\delta}} p_{\varepsilon,\delta,j} \operatorname{div} \phi dx + \int_{S_{ij}} \sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,i}) \cdot \phi ds + \int_{S_{ji}} \sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,j}) \cdot \phi ds = 0. \end{aligned}$$

Here $\sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,i}) = (\nu(\nabla \mathbf{u}_{\varepsilon,\delta} + (\nabla \mathbf{u}_{\varepsilon,\delta})^T) - p_{\varepsilon,\delta,i} I) \mathbf{n} = \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} - p_{\varepsilon,\delta,i} \mathbf{n}$, \mathbf{n} is an outer normal vector with respect to $B_i^{\varepsilon,\delta}$ (respectively, $B_j^{\varepsilon,\delta}$), and I is the identity matrix. Note that $(\mathbf{u}_{\varepsilon,\delta}, \nabla) \mathbf{u}_{\varepsilon,\delta} = 0$ in $B_{ij}^{dec,\varepsilon}$ because $\mathbf{u}_{\varepsilon,\delta}$ is a Womersley function in this cylinder. Taking into account vanishing of terms $\int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta,i} \operatorname{div} \phi dx$, we get

$$\begin{aligned} & \int_{S_{ij}} \left((|e_{ij}| - 2\delta) (u_{\varepsilon,\delta,1,t} - \nu \Delta u_{\varepsilon,\delta,1}) + \sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,i})|_{S_{ij}} + \sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,j})|_{S_{ji}} \right) \cdot \mathbf{e}_{ij} \phi_1 ds \\ & = 0, \text{ where } \mathbf{e}_{ij} \text{ is the vector director of } O_i \bar{O}_j. \end{aligned}$$

On the other hand, the first component of ϕ is an arbitrary function with vanishing mean. So, in every $B_{ij}^{dec,\varepsilon}$ we get an equation for $u_{\varepsilon,\delta,1}$, the first component of $\mathbf{u}_{\varepsilon,\delta}$:

$$(A.20) \quad \begin{aligned} & (|e_{ij}| - 2\delta) (u_{\varepsilon,\delta,1,t} - \nu \Delta u_{\varepsilon,\delta,1}) + (\sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,i})|_{S_{ij}} \\ & + \sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,j})|_{S_{ji}}) \cdot \mathbf{e}_{ij} = D_{ij}(t), \end{aligned}$$

where $D_{ij}(t)$ are “constants” depending on time only.

2. Let us take now an arbitrary test function from the space $\mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$ and again take into account vanishing of terms $\int_{B_i^{\varepsilon,\delta}} p_{\varepsilon,\delta,i} \operatorname{div} \phi dx$; we get

$$(A.21) \quad \sum_{i,j:i < j, \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{S_{ij}} D_{ij}(t) \phi_1 dx = \sum_{i,j:i < j, \overline{O_i O_j} \in \{e_1, \dots, e_M\}} D_{ij} F_{ij} = 0,$$

where $F_{ij}(t) = \int_{S_{ij}} \phi_1 ds = (|e_{ij}| - 2\delta)^{-1} \int_{B_{ij}^{dec,\varepsilon}} \phi_1 dx$.

Varying ϕ we get that for any arbitrary set of fluxes F_{ij} satisfying equation

$$(A.22) \quad \sum_{j: e_{ij} \subset \mathcal{B}} F_{ij} = 0$$

for all $i = 1, \dots, N$, the following relation holds:

$$(A.23) \quad \sum_{j: e_{ij} \subset \mathcal{B}} D_{ij}(t) F_{ij} = 0.$$

3. Consider now an arbitrary function from the space $\mathbf{H}_0^{1,\delta}(B_\varepsilon)$. Recall that such functions satisfy condition (4.4) but are no longer divergence free in $B_i^{\varepsilon,\delta}$. Consider the expression in the assertion of the lemma:

$$\begin{aligned} & \sum_{i=1}^N \sum_{j: \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{\partial B_i^{\varepsilon,\delta} \cap S_{ij}} \left(\nu \frac{\partial \mathbf{u}_{\varepsilon,\delta}}{\partial \mathbf{n}} - p_{\varepsilon,\delta} \mathbf{n} \right) \cdot \phi ds \\ & + \sum_{l=1}^M d_l \int_{\sigma_\varepsilon^{(e_l)}} \mathbf{u}_{\varepsilon,\delta,t} \cdot \phi + \nu \nabla_{x^{(e_l)},t} \mathbf{u}_{\varepsilon,\delta} : \nabla_{x^{(e_l)},t} \phi dx^{(e_l),t}. \end{aligned}$$

Using now relations (A.20), we get that this expression is equal to

$$\sum_{i,j: i < j, \overline{O_i O_j} \in \{e_1, \dots, e_M\}} \int_{S_{ij}} D_{ij}(t) \phi_1 dx = \sum_{i,j: i < j, e_{ij} \subset \mathcal{B}} D_{ij}(t) F_{ij}.$$

Note that test functions satisfy relations (4.4) which implies (A.22), and so relations (A.23) hold.

Lemma A.4 is proved. \square

Remark A.5. Note that in the case $N = M + 1$ condition (4.4) can be removed from the definition of the reduced space, and we can take $D_{ij} = 0$ without condition (A.22), only controlling constants θ_i . Indeed, in (A.20) every stress $\sigma_n(\mathbf{u}_{\varepsilon,\delta}, p_{\varepsilon,\delta,i})|_{S_{ij}}$ is defined up to an additive constant $\theta_i(t)$, so relations (A.20) define uniquely some constants $\mathcal{F}_{ij}(t)$ equal to $\theta_i(t) - \theta_j(t) - D_{ij}(t)$. Consider the following system of equations for θ_i :

$$(A.24) \quad \theta_i(t) - \theta_j(t) = \mathcal{F}_{ij}(t), i < j, e_{ij} \subset \mathcal{B}.$$

Setting $\theta_N = 0$, we get the system of $N - 1$ equations with nondegenerate matrix. Indeed, the homogeneous system is $\theta_i = \theta_j, i < j, e_{ij} \subset \mathcal{B}$. It means that due to the connectedness of the graph \mathcal{B} the unique solution to the homogeneous system is all $\theta_i = 0$. So, system (A.24) admits a unique solution such that $\theta_N = 0$. Consequently, constants θ_i can be chosen in such a way that $D_{ij} = 0$. This choice of θ_i is unique up to one constant θ_N . So, integral identity (4.5) holds true for a wider subspace of test functions with removed condition (4.4).

A.4. Estimate for the difference between the exact solution and the MAPDD solution for the nonstationary Navier–Stokes equations. The result of the previous section can be generalized for the nonstationary problem for the Navier–Stokes equations (2.1) using the approach of [15] and [16].

The proof of the existence of the unique solution to (2.1) for sufficiently small values of ε repeats literally the proof from [15]. A complete asymptotic expansion of order J to this problem is constructed in [15], [16] and it has the form

$$\begin{aligned} \mathbf{v}_\varepsilon^{(J)}(x, t) = & \sum_{i=1}^M \zeta \left(\frac{x_1^{(e_i)}}{3d\varepsilon} \right) \zeta \left(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon} \right) \left(\mathbf{V}_P^{[i, J, NSI]} \left(\frac{x^{(e)'} }{\varepsilon}, t \right) \right. \\ & + \varepsilon^2 \mathbf{V}_P^{[i, J, NSII]} \left(\frac{x^{(e)'} }{\varepsilon}, \frac{t}{\varepsilon^2} \right) \Big) \\ & + \sum_{l=1}^N \left(1 - \zeta \left(\frac{|x - O_l|}{|e|_{\min}} \right) \right) \left(\mathbf{V}^{[BLO_l, J, NSI]} \left(\frac{x - O_l}{\varepsilon}, t \right) \right. \\ & + \varepsilon^2 \mathbf{V}^{[BLO_l, J, NSII]} \left(\frac{x - O_l}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \Big), \end{aligned}$$

where the functions $\mathbf{V}_P^{[i, J, NSI]}$ and $\mathbf{V}_P^{[i, J, NSII]}$ are C^2 -smooth functions such that in local variables only the longitudinal component of the velocity may be different from zero and its longitudinal derivative is equal to zero, while $\mathbf{V}^{[BLO_l, J, NSI]}$ and $\mathbf{V}^{[BLO_l, J, NSII]}$ are the boundary layer correctors belonging to the spaces $L^2((0, T); \mathbf{H}^1(\Omega_l))$ and $L^2((0, \infty); \mathbf{H}^1(\Omega_l))$, respectively, and such that for some positive β , $e^{\beta|y|} \mathbf{V}^{[BLO_l, J, NSI]}$ and $e^{\beta|y|} \mathbf{V}^{[BLO_l, J, NSII]}$ also belong to these spaces. The corresponding norms of these four terms are bounded by constants independent of ε . The vector-function $\mathbf{v}_\varepsilon^{(J)}$ satisfies the estimate

$$\sup_{t \in (0, T)} \|\mathbf{v}_\varepsilon^{(J)} - \mathbf{u}_\varepsilon\|_{L^2(B_\varepsilon)} + \|\nabla \mathbf{v}_\varepsilon^{(J)} - \nabla \mathbf{u}_\varepsilon\|_{L^2((0, T); L^2(B_\varepsilon))} = O(\varepsilon^{J-2}).$$

For more details see [15], [16].

Remark A.6. An asymptotic expansion for the pressure was as well constructed in [15], [16] and has a similar structure:

$$\begin{aligned} p_\varepsilon^{(J)}(x, t) = & \sum_{i=1}^M \zeta \left(\frac{x_1^{(e_i)}}{3d\varepsilon} \right) \zeta \left(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon} \right) \left(p^{[i, J, NSI]}(x_1^{(e)}, t) \right. \\ & + p^{[i, J, NSII]} \left(x_1^{(e)}, \frac{t}{\varepsilon^2} \right) \Big) \\ & + \sum_{l=1}^N \left(1 - \zeta \left(\frac{|x - O_l|}{|e|_{\min}} \right) \right) \left(P^{[BLO_l, J, NSI]} \left(\frac{x - O_l}{\varepsilon}, t \right) \right. \\ & + P^{[BLO_l, J, NSII]} \left(\frac{x - O_l}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \Big), \end{aligned}$$

where the terms $p^{[i, J, NSI]}(x_1^{(e)}, t)$ and $p^{[i, J, NSII]}(x_1^{(e)}, t)$ are linear in $x_1^{(e)}$, where $P^{[BLO_l, J, NSI]}$ and $P^{[BLO_l, J, NSII]}$ are the boundary layer terms belonging to the spaces $L^2((0, T); L^2(\Omega_l))$ and $L^2((0, \infty); L^2(\Omega_l))$, respectively, and such that for some positive β , $e^{\beta|y|} P^{[BLO_l, J, NSI]}$ and $e^{\beta|y|} P^{[BLO_l, J, NSII]}$ also belong to these spaces. (Standard theorems on the asymptotic behavior of the pressure in domains with outlets at infinity establish that it tends to some constants; however, the construction of asymptotic expansion in [15] subtracts these constants so that the pressure terms belong to the space L^2 .) The couple $(\mathbf{v}_\varepsilon^{(J)}, p_\varepsilon^{(J)})$ satisfies the Navier–Stokes equations in the classical sense with a residual of order $O(\varepsilon^{J-2})$ in the $H^1((0, T); \mathbf{L}^2(B_\varepsilon))$ -norm. The boundary and the initial conditions are satisfied exactly.

Consider the following setting: find $\mathbf{v}_{\varepsilon, \delta} \in L^2(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1, \delta}(B_\varepsilon))$ such that $\mathbf{u}_{\varepsilon, \delta} = \mathbf{v}_{\varepsilon, \delta} + \tilde{\mathbf{g}}_\varepsilon$, $\mathbf{v}_{\varepsilon, \delta}(x, 0) = 0$, $\mathbf{v}_{\varepsilon, \delta} \in L^2(0, T; \mathbf{H}_{div0}^{1, \delta}(B_\varepsilon))$, $\mathbf{v}_{\varepsilon, \delta, t} \in L^2(0, T; L^2(B_\varepsilon))$, and $\mathbf{v}_{\varepsilon, \delta}$ satisfies the integral identity

(A.25)

$$\begin{aligned} & \int_{B_\varepsilon} \left(\mathbf{v}_{\varepsilon,\delta,t} \cdot \Phi + \nu \nabla \mathbf{v}_{\varepsilon,\delta} \cdot \nabla \Phi - ((\mathbf{u}_\varepsilon + \tilde{\mathbf{g}}_\varepsilon) \cdot \nabla) \Phi \cdot \mathbf{v}_{\varepsilon,\delta} - (\mathbf{v}_{\varepsilon,\delta} \cdot \nabla) \Phi \cdot \tilde{\mathbf{g}}_\varepsilon \right) dx \\ &= -\nu \int_{B_\varepsilon} \nabla \tilde{\mathbf{g}}_\varepsilon \cdot \nabla \Phi \, dx - \int_{B_\varepsilon} ((\tilde{\mathbf{g}}_\varepsilon \cdot \nabla) \tilde{\mathbf{g}}_\varepsilon + \tilde{\mathbf{g}}_{\varepsilon t}) \cdot \Phi \, dx \end{aligned}$$

for every divergence-free vector-field $\Phi \in \mathbf{H}_{div0}^{1,\delta}(B_\varepsilon)$. Here an extension $\tilde{\mathbf{g}}_\varepsilon$ is such that it belongs to $L^2(0, T; \mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon))$.

Existence and uniqueness of a solution to an analogous problem set in the usual Sobolev spaces is proved in [15] by the Galerkin method. For the problem (A.25) the proof is just the same. Note only that an orthogonal base exists due to the separability of the space $\mathbf{H}_{div0(\partial B_\varepsilon \setminus \gamma_\varepsilon)}^{1,\delta}(B_\varepsilon)$ (as a subspace of a separable space).

Let us now prove Theorem 4.3.

Proof of Theorem 4.3. The idea of the proof is similar to that of Theorem 4.2. Replace $\mathbf{v}_\varepsilon^{(J)}$ by a new approximation,

$$\begin{aligned} \mathbf{v}_\varepsilon^{(J,a)}(x, t) &= \sum_{i=1}^M \zeta \left(\frac{x_1^{(e_i)}}{3d\varepsilon} \right) \zeta \left(\frac{|e_i| - x_1^{(e_i)}}{3d\varepsilon} \right) \left(\mathbf{V}_P^{[i,J,NSI]} \left(\frac{x^{(e)'} }{\varepsilon}, t \right) \right. \\ &\quad \left. + \varepsilon^2 \mathbf{V}_P^{[i,J,NSI]} \left(\frac{x^{(e)'} }{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right) \\ &\quad + \sum_{l=1}^N \left(1 - \zeta \left(\frac{|x - O_l|}{\delta} \right) \right) \left(\mathbf{V}^{[BLO_l,J,NSI]} \left(\frac{x - O_l}{\varepsilon}, t \right) \right. \\ &\quad \left. + \varepsilon^2 \mathbf{V}^{[BLO_l,J,NSI]} \left(\frac{x - O_l}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \right), \end{aligned}$$

where $\delta = C_J \varepsilon |\ln \varepsilon| |e|_{min}$ and C_J is chosen in such a way that

$$\|\mathbf{V}^{[BLO_l,J,NSI]}\|_{L^2((0,T);\mathbf{H}^2(\Omega_{l,\delta/\varepsilon}))} + \|\mathbf{V}_t^{[BLO_l,J,NSI]}\|_{L^2((0,T);\mathbf{H}^2(\Omega_{l,\delta/\varepsilon}))} = O(\varepsilon^{3J+6})$$

and

$$\|\mathbf{V}^{[BLO_l,J,NSI]}\|_{L^2((0,\infty);\mathbf{H}^2(\Omega_{l,\delta/\varepsilon}))} + \|\mathbf{V}_\tau^{[BLO_l,J,NSI]}\|_{L^2((0,\infty);\mathbf{H}^2(\Omega_{l,\delta/\varepsilon}))} = O(\varepsilon^{3J+6}).$$

Namely, $C_J = \frac{9(J+2)}{\beta |e|_{min}}$; see [15, p. 158]. (In fact, the order $3J+6$ is too high for the estimate of Theorem 4.3, and $J+2$ is enough¹.) Then we use the same arguments as in the proof of Theorem 6.2 of [15]: we check that this new approximation $\mathbf{v}_\varepsilon^{(J+2,a)}$ with the approximation $p_\varepsilon^{(J+2,a)}$ modified in the same way satisfies the Navier–Stokes equations in the classical sense with a residual of order $O(\varepsilon^J)$ in the norm $H^1((0, T); L^2(B_\varepsilon))$ and that the velocity approximation satisfies the boundary and initial conditions exactly. In the same way the pair $(\mathbf{v}_\varepsilon^{(J+2,a)}, p_\varepsilon^{(J+2,a)})$ satisfies the problem (A.25) with

¹ Note that in Remark 5.1 of the paper [15] on pp. 158–159 a misprint is found: everywhere $\zeta\left(\frac{|x-O_l||\ln \varepsilon|}{|e|_{min}}\right)$ should be read as $\zeta\left(\frac{|x-O_l|}{\delta}\right)$, where $\delta = C_J \varepsilon |\ln \varepsilon| |e|_{min}$, and respectively inequality $\frac{|x-O_l||\ln \varepsilon|}{C_J |e|_{min}} \geq 1/3$ should be read as $\frac{|x-O_l|}{C_J \varepsilon |\ln \varepsilon| |e|_{min}} \geq 1/3$. This misprint, however, doesn't influence the further reasoning.

the same error. Applying results from the paper [15] (see the inequality (5.7) in the proof of Theorem 5.1 in [15]), we get the estimate

$$\sup_{t \in (0, T)} \|\mathbf{u}_{\varepsilon, \delta} - \mathbf{v}_{\varepsilon}^{(J+2)}\|_{L^2(B_{\varepsilon})} + \|\nabla(\mathbf{u}_{\varepsilon, \delta} - \mathbf{v}_{\varepsilon}^{(J+2)})\|_{L^2((0, T); L^2(B_{\varepsilon}))} = O(\varepsilon^J).$$

Applying the estimate

$$\sup_{t \in (0, T)} \|\mathbf{u}_{\varepsilon} - \mathbf{v}_{\varepsilon}^{(J+2)}\|_{L^2(B_{\varepsilon})} + \|\nabla(\mathbf{u}_{\varepsilon} - \mathbf{v}_{\varepsilon}^{(J+2)})\|_{L^2((0, T); L^2(B_{\varepsilon}))} = O(\varepsilon^J)$$

and the triangle inequality we derive (4.6). \square

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